DSGE Modeling and Statistical Inference

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Introduction

- Estimated dynamic stochastic general equilibrium (DSGE) models are now widely used for
  - empirical research in macroeconomics;
  - quantitative policy analysis and prediction at central banks.
- This mini course will focus on
  - econometric methods to conduct quantitative analysis with DSGE models;
- We will start with a prototypical New Keynesian DSGE model...

Overview

1. A small-scale DSGE model: specification, steady states, log-linearization, first-order approximation to equilibrium dynamics, state-space representation.
2. Statistical inference: frequentist versus Bayesian; use the Kalman filter to evaluate likelihood function.
3. Frequentist inference: maximum likelihood, simulated minimum distance approaches, GMM
4. Bayesian inference: priors, posteriors, Metropolis-Hastings algorithm, post-processing draws.
5. Applications to monetary and fiscal policy analysis.

Prototypical Applications to Monetary and Fiscal Policy Analysis

1. What is the optimal target inflation rate?
2. Was high inflation and output volatility in the 1970s due to loose monetary policy?
3. Effects of the zero lower bound on nominal interest rates on monetary policy.
4. How large are government spending multipliers?

A Small-Scale New Keynesian DSGE Model

The model consists of

- households;
- final goods producing firms;
- intermediate goods producing firms;
- central bank and fiscal authority;
- exogenous shock processes

Households

- Households maximize

$$\mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \beta^{(t-\tau)} \left\{ \ln C_t - \frac{\phi_t}{1+\nu} L_t^{1+\nu} \right\} \right]$$

- subject to the constraints:

$$P_t C_t + B_{t+1} \leq P_t W_t L_t + \Pi_t + R_{t-1} B_t - T_t + \Omega_t.$$

- In a nutshell:
  - household cares about the future: intertemporal optimization
  - household likes consumption
  - household does not like to work...
  - there is a budget constraint: can't spend more than you earn and borrow; have to pay taxes;

Households: First-Order Conditions

- Households maximize

$$\mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \beta^{(t-\tau)} \left\{ \ln C_t - \frac{\phi_t}{1+\nu} L_t^{1+\nu} \right\} \right]$$

- subject to the constraints:

$$P_t C_t + B_{t+1} \leq P_t W_t L_t + \Pi_t + R_{t-1} B_t - T_t + \Omega_t.$$

- Introduce Lagrange multiplier μ_t for budget constraint.
- Lagrangian

$$\begin{aligned} \mathcal{L} &= \mathbb{E}_\tau \Bigg[\sum_{t=\tau}^{\infty} \beta^{(t-\tau)} \Bigg\{ \ln C_t - \frac{\phi_t}{1+\nu} L_t^{1+\nu} \\ & \quad -\mu_t \Big(P_t C_t + B_{t+1} - \big[P_t W_t L_t + \Pi_t + R_{t-1} B_t - T_t + \Omega_t \big] \Big) \Bigg\} \Bigg] \end{aligned}$$

Households: First-Order Conditions

- First-order condition for C_t:

$$\frac{1}{C_t} = \mu_t P_t$$

- First-order condition for B_{t+1}:

$$\mu_t = \beta \mathbb{E}_t[\mu_{t+1} R_t]$$

- Combine to consumption Euler equation (define $\pi_{t+1} = P_{t+1}/P_t$):

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{R_t}{\pi_{t+1}} \right]$$

- Labor supply – first-order condition for L_t:

$$\phi_t L_t^{\nu} = \mu_t P_t W_t = \frac{W_t}{C_t}.$$

A Small-Scale New Keynesian DSGE Model

- households;
- **final goods producing firms;**
- intermediate goods producing firms;
- central bank and fiscal authority;
- exogenous shock processes

Final Goods Production

- Production: (these guys just buy and combine intermediate goods)

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\lambda_t}} di \right]^{1+\lambda_t}$$

- Profits

$$Y_t P_t - \int Y_t(i) P_t(i) di = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\lambda_t}} di \right]^{1+\lambda_t} P_t - \int Y_t(i) P_t(i) di.$$

- Take prices as given and maximize profits by choosing optimal inputs $Y_t(i)$:

$$P_t(i) = P_t Y_t^{\lambda_t/(1+\lambda_t)} Y_t(i)^{-\lambda_t/(1+\lambda_t)} \quad \Longrightarrow \quad Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\frac{1+\lambda_t}{\lambda_t}} Y_t$$

- Free entry leads to zero profits:

$$Y_t P_t = \int Y_t(i) P_t(i) di \quad \Longrightarrow \quad P_t = \left[\int_0^1 P_t(i)^{-\frac{1}{\lambda_t}} di \right]^{-\lambda_t}.$$

- Aggregate inflation is defined as $\pi_t = P_t / P_{t-1}$.

A Small-Scale New Keynesian DSGE Model

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- exogenous shock processes

Intermediate Goods Production

- Production (these guys hire to produce something):

$$Y_t(i) = \max \left\{ A_t L_t(i) - \mathcal{F}, 0 \right\}.$$

- Firms are monopolistically competitive; face downward sloping demand curve:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\frac{1+\lambda_t}{\lambda_t}} Y_t.$$

- Firms set prices to maximize profits, but there is a friction:
  - firms can only re-optimize their prices with probability $1 - \zeta_p$;
  - remaining $1 - \iota$ firms adjust their prices by $\bar{\pi}$
- Once prices are set, firms have to produce whatever quantity is demanded.

Intermediate Goods Production

- Define the real marginal costs of producing a unit Y_{it} as

$$MC_t = \frac{W_t}{A_t}$$

- Decision problem ($\beta^s \Xi_{t+s|t}$ is today's value of a future dollar)

$$\max_{\tilde{P}_t(i)} \quad \mathbb{E}_t \left\{ \sum_{s=0}^{\infty} \zeta_p^s \beta^s \Xi_{t+s|t} Y_{t+s}(i) \left[\tilde{P}_t(i) \bar{\pi}^s - P_{t+s} MC_{t+s} \right] \right\}$$

$$\text{s.t.} \quad Y_{t+s}(i) = \left(\frac{\tilde{P}_t(i) \bar{\pi}^s}{P_{t+s}} \right)^{-\frac{1+\lambda_t}{\lambda_t}} Y_{t+s}$$

- Differentiate with respect to $\tilde{P}_t(i)$ to obtain first-order condition for optimal price.

A Small-Scale New Keynesian DSGE Model

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Monetary Policy

- We did not specify a money demand equation, but we could. It would depend on the nominal interest rate. The higher R_t, the lower the demand for money.
- Central bank prints enough money so that demand is satisfied at interest rate implied by monetary policy rule:

$$R_t = R_{*,t}^{1-\rho_R} R_{t-1}^{\rho_R} \exp\{\sigma_R \epsilon_{R,t}\}, \quad R_{*,t} = (r\pi_*) \left(\frac{\pi_t}{\pi_*}\right)^{\psi_1} \left(\frac{Y_t}{\gamma Y_{t-1}}\right)^{\psi_2}$$

- r is equilibrium real rate.
- π_* is target inflation rate.
- $\epsilon_{R,t}$ is exogenous monetary policy shock. Interpretation?

Fiscal Policy

- For now, it's passive and not very interesting.
- Budget constraint:

$$P_t G_t + R_{t-1} B_t + M_t = T_t + B_t + M_{t+1}$$

- Lump-sum taxes/transfer balance the budget in every period. Seigniorage does not matter.
- Government spending is exogenous. Re-scale:

$$G_t = \left(1 - \frac{1}{g_t}\right) Y_t.$$

A Small-Scale New Keynesian DSGE Model

- households;
- final goods producing firms;
- intermediate goods producing firms;
- central bank and fiscal authority;
- **exogenous shock processes.**

Exogenous shock processes

- Total factor productivity A_t.
- Preference / labor demand shifter ϕ_t.
- Mark-up shock λ_t.
- Monetary policy shock $\epsilon_{R,t}$.
- Government spending shock g_t.
- We will specify exogenous laws of motions for these processes, e.g.,

$$\ln g_t = (1-\rho_g)\ln g^* + \rho_g \ln g_{t-1} + \sigma_g \epsilon_{g,t}, \quad \epsilon_{g,t} \sim N(0,1).$$

A Small-Scale New Keynesian DSGE Model

So far:

- households;
- final goods producing firms;
- intermediate goods producing firms;
- central bank and fiscal authority;
- exogenous shock processes.

What's left to do?

- derive aggregate resource constraint;
- derive evolution of price dispersion;
- complete markets assumption;

Model Solution

- After deriving the equilibrium conditions of the model, we now need to solve for the dynamics of the endogenous variables.
- System of nonlinear expectational difference equations;
- Find solution(s) of system of expectational difference equations:
  - global (nonlinear) approximation methods;
  - local approximation near steady state.
- We will focus on log-linear approximations around the steady state.
- Many more details in FVRRS.

Our Goal: State-space Representation of DSGE Model

- $n_y \times 1$ vector of observables:

$$y_t = M_y'[\log(X_t/X_{t-1}), \log lsh_t, \log \pi_t, \log R_t]'.$$

- $n_s \times 1$ vector of econometric state variables s_t

$$s_t = [\phi_t, \lambda_t, z_t, \epsilon_{R,t}, \widehat{x}_{t-1}]'$$

- DSGE model parameters:

$$\theta = [\beta, \gamma, \lambda, \pi^*, \zeta_p, \nu, \rho_\phi, \rho_\lambda, \rho_z, \sigma_\phi, \sigma_\lambda, \sigma_z, \sigma_R]'.$$

- Measurement equation:

$$y_t = \Psi_0(\theta) + \Psi_1(\theta)s_t.$$

- State-transition equation:

$$s_t = \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t, \quad \epsilon_t = [\epsilon_{\phi,t}, \epsilon_{\lambda,t}, \epsilon_{z,t}, \epsilon_{R,t}]'$$

Our Goal: State-Space Representation of DSGE Model

State-space representation:

$$
\begin{aligned}
y_t &= \Psi_0(\theta) + \Psi_1(\theta) s_t \\
s_t &= \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t
\end{aligned}
$$

System matrices:

$$
\Psi_0(\theta) = M_y' \begin{bmatrix} \log \gamma \\ \log(lsh) \\ \log \pi^* \\ \log(\pi^* \gamma/\beta) \end{bmatrix}, \quad x_\phi = -\frac{\kappa_p \psi_p/\beta}{1-\psi_p\rho_\phi}, \quad x_\lambda = -\frac{\kappa_p \psi_p/\beta}{1-\psi_p\rho_\lambda}, \quad x_z = \frac{\rho_z \psi_p}{1-\psi_p \rho_z}, \quad x_{\epsilon_R} = -\psi_p \sigma_R
$$

$$
\Psi_1(\theta) = M_y' \begin{bmatrix} x_\phi & x_\lambda & x_z + 1 & x_{\epsilon_R} & -1 \\ 1+(1+\nu)x_\phi & (1+\nu)x_\lambda & (1+\nu)x_z & (1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p}{1-\beta\rho_z}(1+\nu)x_z & +\kappa_p(1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p/\beta}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p/\beta}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p/\beta}{1-\beta\rho_z}(1+\nu)x_z & (\kappa_p(1+\nu)x_{\epsilon_R}/\beta + \sigma_R) & 0 \end{bmatrix}
$$

$$
\Phi_1(\theta) = \begin{bmatrix} \rho_\phi & 0 & 0 & 0 & 0 \\ 0 & \rho_\lambda & 0 & 0 & 0 \\ 0 & 0 & \rho_z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_\phi & x_\lambda & x_z & x_{\epsilon_R} & 0 \end{bmatrix}, \quad \Phi_\epsilon(\theta) = \begin{bmatrix} \sigma_\phi & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

M_y' is an $n_y \times 4$ selection matrix that selects rows of Ψ_0 and Ψ_1.

Steady State

Shut down aggregate uncertainty: set all shock standard deviations $\sigma_{\cdot} = 0$.

- **Technology:**

$$\ln A_t = \ln \gamma + \ln A_{t-1} + z_t, \quad z_t = \rho_z z_{t-1} + \sigma_z \epsilon_{z,t}.$$

Set $\sigma_z = 0$: $\ln A_t^* = \gamma t$.

- **Preferences:**

$$\ln \phi_t = (1 - \rho_\phi) \ln \phi + \rho_\phi \ln \phi_{t-1} + \sigma_\phi \epsilon_{\phi,t}.$$

- **Mark-up:**

$$\ln \lambda_t = (1 - \rho_\lambda) \ln \lambda + \rho_\lambda \ln \lambda_{t-1} + \sigma_\lambda \epsilon_{\lambda,t}.$$

- **Government Spending:**

$$\ln g_t = (1 - \rho_g) \ln g^* + \rho_g \ln g_{t-1} + \sigma_g \epsilon_{g,t}$$

Steady State

- Problem: this economy grows... which does not lead to a **steady** state.
- Solution: detrend model variables by A_t.
- Model has steady state in terms of detrended variables.

Example: Detrend Households' Euler Equation

- Recall:

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{R_t}{\pi_{t+1}} \right]$$

- Rewrite:

$$\frac{A_t}{C_t} = \beta \mathbb{E}_t \left[\frac{A_{t+1}}{C_{t+1}} \frac{A_t}{A_{t+1}} \frac{R_t}{\pi_{t+1}} \right] \quad \Longrightarrow \quad \frac{1}{c_t} = \beta \mathbb{E}_t \left[\frac{1}{c_{t+1}} \frac{1}{\gamma e^{z_{t+1}}} \frac{R_t}{\pi_{t+1}} \right]$$

- Steady state:

$$R = \pi \frac{\gamma}{\beta} = \pi r.$$

Local Approximation Around Steady State

- We will now approximate the equilibrium dynamics of the model.
- Taylor series expansion around around the steady state.

What is a Log-Linear Approximation?

- Consider Cobb-Douglas production function: $Y_t = Z_t K_t^\alpha H_t^{1-\alpha}$.
- Linearization around Y_*, Z_*, K_*, H_*:

$$\begin{aligned} Y_t - Y_* &= K_*^\alpha H_*^{1-\alpha}(Z_t - Z_*) + \alpha Z_* K_*^{\alpha-1} H_*^{1-\alpha}(K_t - K_*) \\ &\quad +(1-\alpha) Z_* K_*^\alpha H_*^{-\alpha}(H_t - K_*) \end{aligned}$$

- Log-linearization: Let $f(x) = f(e^v)$ and linearize with respect to v:

$$f(e^v) \approx f(e^{v_*}) + e^{v_*} f'(e^{v_*})(v - v_*).$$

Thus:

$$f(x) \approx f(x_*) + x_* f'(x_*)(\ln x/x_*) = f(x_*) + f'(x_*)\hat{x}$$

- Cobb-Douglas production function:

$$\tilde{Y}_t = \hat{Z}_t + \alpha \hat{K}_t + (1-\alpha)\hat{H}_t$$

Let's Try the Log-linearizations

- **Euler Equation:**

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left[\frac{1}{c_{t+1}} \frac{1}{\gamma e^{z_{t+1}}} \frac{R_t}{\pi_{t+1}} \right].$$

- **Log-linearized:**

$$-\widehat{c}_t = \mathbb{E}_t \left[-\widehat{c}_{t+1} - z_{t+1} + \widehat{R}_t - \widehat{\pi}_{t+1} \right] \quad \Longrightarrow \quad \widehat{c}_t = \mathbb{E}_t[\widehat{c}_{t+1}] - (\widehat{R}_t - \mathbb{E}[\widehat{\pi}_{t+1}]) + \mathbb{E}_t[z_{t+1}].$$

- **Labor Supply:**

$$\phi_t L_t^{\nu} = \frac{w_t}{c_t}.$$

- **Log-linearized:**

$$\widehat{\phi}_t + \nu \widehat{L}_t = \widehat{w}_t - \widehat{c}_t$$

Combining Bits and Pieces

- Notation: write x_t instead of y_t for output.
- Assume: $\pi = \bar{\pi} = \pi_*$, $\psi_1 = 1/\beta$, $\psi_2 = 0$, $\rho_R = 0$.
- **Linear rational expectations (LRE) system:**

$$\begin{aligned}
\widehat{c}_t &= \mathbb{E}_{t+1}[\widehat{c}_{t+1}] - \left(\widehat{R}_t - \mathbb{E}_t[\widehat{\pi}_{t+1}]\right) + \mathbb{E}_t[z_{t+1}] \\
\widehat{\pi}_t &= \beta\mathbb{E}_t[\widehat{\pi}_{t+1}] + \kappa_p\left(\widehat{lsh}_t + \lambda_t\right) \\
\widehat{R}_t &= \frac{1}{\beta}\widehat{\pi}_t + \sigma_R\epsilon_{R,t} \\
\widehat{lsh}_t &= (1+\nu)\widehat{c}_t + \nu\widehat{g}_t + \phi_t \\
\widehat{x}_t &= \widehat{c}_t + \widehat{g}_t \\
\widehat{g}_t &= \rho_g\widehat{g}_{t-1} + \sigma_g\epsilon_{g,t} \\
\phi_t &= \rho_\phi\phi_{t-1} + \sigma_\phi\epsilon_{\phi,t} \\
\lambda_t &= \rho_\lambda\lambda_{t-1} + \sigma_\lambda\epsilon_{\lambda,t} \\
z_t &= \rho_z z_{t-1} + \sigma_z\epsilon_{z,t}
\end{aligned}$$

How Can One Solve LRE Systems? A Simple Example

Simple model:

$$y_t = \frac{1}{\theta}\mathbb{E}_t[y_{t+1}] + \epsilon_t, \quad \epsilon_t \sim iid(0,1), \quad \theta \in \Theta = [0,2].$$

Sims (2002) Method: Introduce conditional expectation $\xi_t = \mathbb{E}_t[y_{t+1}]$ and forecast error $\eta_t = y_t - \xi_{t-1}$:

$$\xi_t = \theta\xi_{t-1} - \theta\epsilon_t + \theta\eta_t.$$

Nonexplosive solutions:

- Determinacy: $\theta > 1$. The only stable solution:

$$\xi_t = 0, \quad \eta_t = \epsilon_t \quad \Longrightarrow \quad y_t = \epsilon_t$$

- Indeterminacy: $\theta \leq 1$ the stability requirement imposes no restrictions on forecast error:

$$\eta_t = \widetilde{M}\epsilon_t + \zeta_t \quad \Longrightarrow y_t = \theta y_{t-1} + \widetilde{M}\epsilon_t + \zeta_t - \theta\epsilon_{t-1}$$

Solving the LRE System

- A simplified version of our DSGE model can be solved “by hand.”
- More realistic models need to be solved numerically.
- The numerical solution can be expressed as a VAR(1) in terms of suitably chosen model variables s_t:

$$s_t = \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t.$$

- Many solution techniques are available for LRE models, e.g., Blanchard and Kahn (1980), King and Watson (1998), Uhlig (1999), Anderson (2000), Klein (2000), Christiano (2002).

Linking Model Variables and Observables

- To confront the model with data, one has to account for the presence of the model-implied stochastic trend in aggregate output and to add the steady states to all model variables.

- Measurement equations:

$$
\begin{aligned}
\log(X_t/X_{t-1}) &= \widehat{x}_t - \widehat{x}_{t-1} + z_t + \log\gamma \\
\log(lsh_t) &= \widehat{lsh}_t + \log(lsh) \\
\log\pi_t &= \widehat{\pi}_t + \log\pi^* \\
\log R_t &= \widehat{R}_t + \log(\pi^*\gamma/\beta).
\end{aligned}
$$

State-space Representation of DSGE Model

- $n_y \times 1$ vector of observables:

$$y_t = M_y'[\log(X_t/X_{t-1}), \log lsh_t, \log \pi_t, \log R_t]'.$$

- $n_s \times 1$ vector of econometric state variables s_t

$$s_t = [\phi_t, \lambda_t, z_t, \epsilon_{R,t}, \widehat{x}_{t-1}]'$$

- DSGE model parameters:

$$\theta = [\beta, \gamma, \lambda, \pi^*, \zeta_p, \nu, \rho_\phi, \rho_\lambda, \rho_z, \sigma_\phi, \sigma_\lambda, \sigma_z, \sigma_R]'.$$

- Measurement equation:

$$y_t = \Psi_0(\theta) + \Psi_1(\theta)s_t.$$

- State-transition equation:

$$s_t = \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t, \quad \epsilon_t = [\epsilon_{\phi,t}, \epsilon_{\lambda,t}, \epsilon_{z,t}, \epsilon_{R,t}]'$$

State-Space Representation of DSGE Model

State-space representation:

$$
\begin{aligned}
y_t &= \Psi_0(\theta) + \Psi_1(\theta) s_t \\
s_t &= \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t
\end{aligned}
$$

System matrices:

$$
\Psi_0(\theta) = M_y' \begin{bmatrix} \log\gamma \\ \log(lsh) \\ \log\pi^* \\ \log(\pi^*\gamma/\beta) \end{bmatrix}, \quad x_\phi = -\frac{\kappa_p \psi_p/\beta}{1-\psi_p\rho_\phi}, \quad x_\lambda = -\frac{\kappa_p\psi_p/\beta}{1-\psi_p\rho_\lambda}, \quad x_z = \frac{\rho_z \psi_p}{1-\psi_p\rho_z}, \quad x_{\epsilon_R} = -\psi_p\sigma_R
$$

$$
\Psi_1(\theta) = M_y' \begin{bmatrix} x_\phi & x_\lambda & x_z + 1 & x_{\epsilon_R} & -1 \\ 1+(1+\nu)x_\phi & (1+\nu)x_\lambda & (1+\nu)x_z & (1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p}{1-\beta\rho_z}(1+\nu)x_z & +\kappa_p(1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p/\beta}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p/\beta}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p/\beta}{1-\beta\rho_z}(1+\nu)x_z & (\kappa_p(1+\nu)x_{\epsilon_R}/\beta + \sigma_R) & 0 \end{bmatrix}
$$

$$
\Phi_1(\theta) = \begin{bmatrix} \rho_\phi & 0 & 0 & 0 & 0 \\ 0 & \rho_\lambda & 0 & 0 & 0 \\ 0 & 0 & \rho_z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_\phi & x_\lambda & x_z & x_{\epsilon_R} & 0 \end{bmatrix}, \quad \Phi_\epsilon(\theta) = \begin{bmatrix} \sigma_\phi & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

M_y' is an $n_y \times 4$ selection matrix that selects rows of Ψ_0 and Ψ_1.

Statistical Inference

- Simulation: pick values for parameter vector $\theta \Longrightarrow$ simulate data $Y^{sim}(\theta)$ and determine its properties.
- Statistical inference:
  - observed data $Y^{obs} \Longrightarrow$ determine suitable values for parameter vector θ.
  - **Basic Idea:** choose θ such that $Y^{sim}(\theta)$ look like Y^{obs}.
  - Goals: estimates $\hat{\theta}$ as well as measures of uncertainty associated with these estimates.

Statistical Inference

- Frequentist:
  - pre-experimental perspective;
  - condition on "true" but unknown θ_0;
  - treat data Y as random;
  - study behavior of estimators and decision rules under repeated sampling.
- Bayesian:
  - post-experimental perspective;
  - condition on observed sample Y;
  - treat parameter θ as unknown and random;
  - derive estimators and decision rules that minimize expected loss (averaging over θ) conditional on observed Y.

Frequentist Inference

DSGE model (M_1) is assumed to be correctly specified, i.e. we believe the probabilistic structure is rich enough to assign high probability to the salient features of macroeconomic time series.

- Desirable to let the model-implied probability distribution $p(Y|\theta_0, M_1)$ determine the choice of the objective function for estimators and test statistics to obtain a statistical procedure that is efficient (meaning that the estimator is close to θ_0 with high probability in repeated sampling).

- Maximum likelihood (ML) estimator

$$\hat{\theta}_{ml} = \text{argmax}_{\theta \in \Theta} \log p(Y|\theta, M_1).$$

- Minimize discrepancy between sample statistics $\hat{m}_T(Y)$ and model-implied population statistics $\mathbb{E}[\hat{m}_T(Y)|\theta, M_1]$:

$$\hat{\theta}_{md} = \text{argmin}_{\theta \in \Theta} Q_T(\theta|Y) = \left\| \hat{m}_T(Y) - \mathbb{E}[\hat{m}_T(Y)|\theta, M_1] \right\|_{W_T},$$

Bayesian Inference

DSGE model (M_1) is assumed to be correctly specified, i.e. we believe the probabilistic structure is rich enough to assign high probability to the salient features of macroeconomic time series.

- Initial state of knowledge summarized in **prior** distribution $p(\theta)$.
- Update in view of data Y to obtain **posterior** distribution $p(\theta|Y)$:

$$p(\theta|Y,M_1) = \frac{p(Y|\theta,M_1)p(\theta|M_1)}{p(Y|M_1)}, \quad p(Y|M_1) = \int p(Y|\theta,M_1)p(\theta|M_1)d\theta.$$

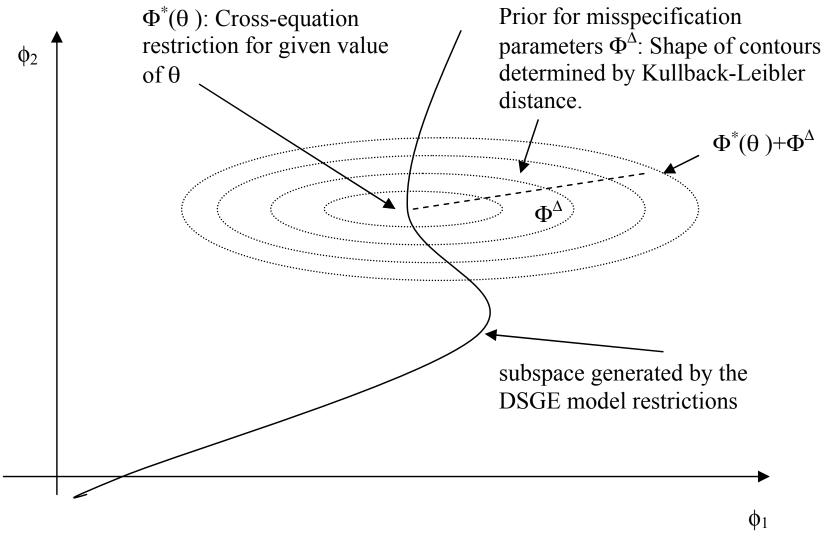
- Make decisions that minimize posterior expected loss:

$$\delta_* = \mathrm{argmin}_{\delta \in \mathcal{D}} \int L\big(h(\theta),\delta\big)p(\theta|Y,M_1)d\theta.$$

- Place probabilities on competing models and update:

$$\frac{\pi_{1,T}}{\pi_{2,T}} = \frac{\pi_{1,0}}{\pi_{2,0}} \frac{p(Y|M_1)}{p(Y|M_2)}.$$

Model Misspecification is a Concern



Frequentist Inference

DSGE model (M_1) is assumed to be misspecified or incompletely specified.

- Example: suppose our DSGE model only has a monetary policy shock. Then,

$$\frac{1}{\kappa_p(1+\nu)x_{\epsilon_R}/\beta+\sigma_R}\widehat{R}_t - \frac{1}{\kappa_p(1+\nu)x_{\epsilon_R}}\widehat{\pi}_t = 0,$$

  which is clearly violated in the data.
- Need reference model M_0, e.g., VAR, under which to evaluate sampling distribution of Y.
- Concept of "true" value is no longer sensible $\Longrightarrow$ pseudo-optimal parameter value:

$$\theta_0(Q,W) = \text{argmin}_{\theta\in\Theta}\ Q(\theta|M_0),$$

  where

$$Q(\theta|M_0) = \Big\|\mathbb{E}[\hat{m}_T(Y)|M_0] - \mathbb{E}[\hat{m}(Y)|\theta, M_1]\Big\|_W.$$

DSGE model (M_1) is assumed to be misspecified or incompletely specified.

- Derive posterior distributions under a more flexible reference model M_0, e.g., VAR. Then choose θ to minimize discrepancy between implications of M_0 and DSGE model M_1.

- Use DSGE model M_1 to generate a prior distribution for a more flexible reference model M_0.
- Rather than using posterior probabilities to select among or average across two DSGE models, one can form a prediction pool, which is essentially a linear combination of two predictive densities:

$$\lambda p(y_t|Y_{1:t-1}, M_1) + (1-\lambda)p(y_t|Y_{1:t-1}, M_2).$$

The weight $\lambda \in [0,1]$ can be determined based on

$$\prod_{t=1}^{T} \left[\lambda p(y_t|Y_{1:t-1}, M_1) + (1-\lambda)p(y_t|Y_{1:t-1}, M_2)\right].$$