Individual Behaviors

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Standard disclaimers apply.

The Promise

- “The latter (HJB equation) is much better for numerical approximations”
- Somewhat hard to explain intuitively...
- Feel free to stop me any time, but read the codes in your own time again with the slides.

HJB Equations

$$\rho V(x) = \max_a \left[u(x,a) + f(x,a) \frac{\mathrm{d}V}{\mathrm{d}x} + \frac{g(x,a)^2}{2} \frac{\mathrm{d}^2 V}{\mathrm{d}x^2} \right]$$

HJB Equations

- HJB equations rarely have closed for solutions, so need to be solved numerically
- Discretize and solved the discretized system.
  - Need to know that the discretized solution converges to the true solution.

Barles-Souganidis

- There is a very general framework on discretization (Barles and Souganidis, 1991)[1]
- A discretization converges to the unique continuous viscosity solution if the discretization satisfies
  1. stability
  2. consistency
  3. monotonicity
  4. strong uniqueness
- Most requires are automatic
- Only the *monotonicity* requires care
  - Diffusion term is also automatic for monotonicity
  - Drift terms are the one that requires care

[1] Barles, G., & Souganidis, P. E. (1991). Convergence of approximation schemes for fully nonlinear second order equations. Asymptotic analysis, 4(3), 271-283.

Barles-Souganidis: Monotonicity

- Intuition: Translation of the contraction property of value functions into discretization
  - Want to prevent
    1. Value function at savings of \$100 is not high enough because the value at \$200 is too high
    2. ($\Rightarrow$) increase value function at \$100
    3. Value function at \$200 is not high enough because the value at \$100 is too high
    4. ($\Rightarrow$) increase value function at \$200
    5. repeat
  - ($\Rightarrow$) Only allow the impact of the value functions at other points to move the discretization equation in the same direction[2]
  - In practice, one can follow "positive coefficient rule" of putting positive weights.
  - We will come back to this.

[2] Disclaimer: This is just an analogy/intuition, so is not perfect.

HJB Equations

- HJB equations rarely have closed for solutions, so need to be solved numerically
- Discretize and solved the discretized system.
  - Need to know that the discretized solution converges to the true solution.
  - Need a reasonable representation inside computers

Finite Difference Discretization

- Approximate the system with finite number of grid points. $V(x_i)$ at $\{x_i\}$
- Approximate differential operators with the finite difference operators

$$
\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x_i} &= \frac{V(x_{i'}) - V(x_i)}{x_{i'} - x_i} \\
\left.\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right|_{x_i} &= 2\frac{V(x_{i'})}{(x_{i'} - x_i)(x_{i'} - x_{i''})} \\
&\quad + 2\frac{V(x_i)}{(x_{i'} - x_i)(x_{i''} - x_i)} \\
&\quad - 2\frac{V(x_{i''})}{(x_{i''} - x_i)(x_{i'} - x_{i''})}
\end{aligned}
$$

where $x_{i'}$ and $x_{i''}$ are “nearby” points

Finite Difference Discretization

- HJB equation

$$\rho V(x) = \max_a \left[u(x,a) + f(x,a)\frac{\mathrm{d}V}{\mathrm{d}x} + \frac{g(x,a)^2}{2}\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} \right] \qquad \forall x \in \Omega$$

- turns into

$$\rho V(x_i) = \max_a \left[u(x_i,a) + f(x_i,a)\left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x_i} + \frac{g(x_i,a)^2}{2}\left.\frac{\mathrm{d}^2 V}{\mathrm{d}x^2}\right|_{x_i} \right]$$

$$\forall \{x_i\}$$

Finite Difference Discretization

$$\rho V(x_i) = \max_a \left[u(x_i, a) + f(x_i, a) \left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x_i} + \frac{g(x_i, a)^2}{2} \left.\frac{\mathrm{d}^2 V}{\mathrm{d}x^2}\right|_{x_i} \right]$$

$$\forall \{x_i\}$$

- Recall: the discretization has to satisfy the requirements of Barles-Souganidis
- Only monotonicity requirement requires care

Finite Difference Discretization

$$\rho V(x_i) = \max_a \left[u(x_i, a) + f(x_i, a) \left. \frac{\mathrm{d}V}{\mathrm{d}x} \right|_{x_i} + \frac{g(x_i, a)^2}{2} \left. \frac{\mathrm{d}^2 V}{\mathrm{d}x^2} \right|_{x_i} \right]$$

$$\forall \{x_i\}$$

- Note the optimal action of a^* depends on $\{V_i\}$
- Hence, the above equation is in fact a nasty non-linear equation.
- We usually solve it using iterative method
  - and we want each iteration to be fast

Finite Difference Discretization

$$\rho V^{(n+1)}(x_i) = \left[u(x_i, a^*(V^{(n)})) + f(x_i, a^*(V^{(n)})) \left. \frac{\mathrm{d}V^{(n+1)}}{\mathrm{d}x} \right|_{x_i} \right.$$
$$\left. + \frac{g(x_i, a^*(V^{(n)}))^2}{2} \left. \frac{\mathrm{d}^2 V^{(n+1)}}{\mathrm{d}x^2} \right|_{x_i} \right]$$
$$\forall \{x_i\}$$

- i.e., use solve for the optimal action a^* given the current guess of $V^{(n)}$, and update to get $V^{(n+1)}$
- This is also good because the equation is linear in $V^{(n+1)}$, i.e., solving for $V^{(n+1)}$ is fast!

Finite Difference Discretization

To control the speed of adjustment, we add a term that is zero in steady state

$$
\begin{aligned}
\rho V^{(n+1)}(x_i) = & \left[u(x_i, a^*(V^{(n)})) + f(x_i, a^*(V^{(n)})) \left. \frac{\mathrm{d}V^{(n+1)}}{\mathrm{d}x} \right|_{x_i} \right. \\
& \left. + \frac{g(x_i, a^*(V^{(n)}))^2}{2} \left. \frac{\mathrm{d}^2 V^{(n+1)}}{\mathrm{d}x^2} \right|_{x_i} \right] \\
& - \frac{V^{(n+1)}}{\Delta} + \frac{V^{(n)}}{\Delta} \\
& \forall \{x_i\}
\end{aligned}
$$

which also has an intuitive motivation of "backward" time-stepping.