Individual Behaviors to Aggregate

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Standard disclaimers apply.

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Distribution $\Rightarrow$ Macroeconomy

- So far, we have focused on how to solve the optimization problems of individuals.
- For the heterogeneous agent model to be "closed," we need their behaviors to affect the macroeconomy.

Distribution $\Rightarrow$ Macroeconomy

- This requires computation of the aggregate quantities from individual behaviors.
  - For example, how do the total amount of savings change if you change interest rate.
- i.e., we need things like

$$\sum f(x_i) \qquad\qquad i \in \text{population}$$

- where x_i changes consistently with individual behaviors

Computation of the Distribution

- Wish: $g(x)$ a distribution function such that

$$\frac{1}{N}\sum f(x_i) = \int f(x)g(d)\,\mathrm{d}x$$

- We proceed assume that such a $g(\cdot)$ exists, and get the conditions it need to satisfy
- This results in the Fokker-Planck Equation

Computation of the Distribution

- For the dynamic decision of an individual, we model it as an Ito process

$$\mathrm{d}x_i = \mu(x_i)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

- This is the optimal behavior determined by the HJB equation, but at this state.

Monte Carlo Simulation

- Since we want to approximate

$$\frac{1}{N}\sum f(x_i)$$

  we can in fact, just simulate this directly.

- Hence, we take sample of $\{x_i\}$ of the population.
- At each time step, simulate via

$$x_{i,t+\Delta t} = \mu(x_{i,t})\Delta t + \sigma\varepsilon \qquad\qquad \varepsilon \sim N(0, var = \Delta t)$$

- This is called the Euler-Maruyama scheme.
- As $N \to \infty$, it goes to the correct value. But...

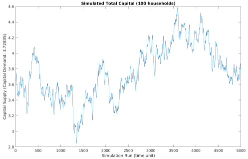
Monte Carlo Simulation

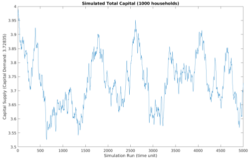
- Size of N?
- Burn-in?
- Step-size: Δt

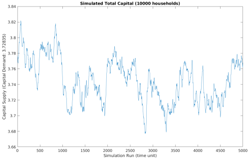
Monte Carlo Simulation

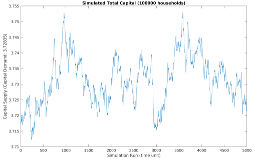
- If one has an experience with the Bayesian estimation, one knows how "annoying" the hyperparameter tuning is.
- Also, based on some experiments, the size of N can be quite large.

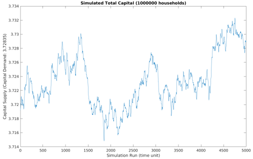
Monte Carlo Simulation











- One need quite a lot of simulation households for the computation of the steady-state distribution of the Bewley-Huggett-Aiyagari model.

[0]Preliminary plot: need to be checked further.

Monte Carlo Simulation

- Some of these problems are gone with the partial differential equations formulation.
- Things scale differently between different methods
  - We might have to come back to the Monte Carlo simulations in higher dimensions, but for low dimensional problems, PDE methods work better.

Fokker-Planck Equation

- Now, to get the partial differential equation formulation, recall we want

$$\frac{1}{N}\sum f(x_i) = \int f(x)g(x)\,\mathrm{d}x$$

Fokker-Planck Equation

- We will consider how $g(\cdot)$ evolves over time as people follow their optimal decisions. Hence, let $g_t(x)$ be the time dependent distribution, i.e.,

$$\frac{1}{N}\sum f(x_{i,t}) = \int f(x)g_t(x)\,\mathrm{d}x$$

- Again, we will do Δt approximation.

Fokker-Planck Equation

$$\frac{1}{N}\sum f(x_{i,t+\Delta t}) = \frac{1}{N}\sum \Big[f(x_{i,t}) + f'(x_{i,t})(x_{i,t+\Delta t} - x_{i,t}) \\ + \frac{1}{2}f''(x_{i,t})(x_{i,t+\Delta t} - x_{i,t})^2\Big]$$

The detail of $(x_{i,t+\Delta t} - x_{i,t})$ is the same as that with HJB equation, so we get

$$\frac{1}{N}\sum f(x_{i,t+\Delta t}) - \frac{1}{N}\sum f(x_{i,t}) = \Delta t\frac{1}{N}\sum f'(x_{i,t})\mu(x_{i,t}) \\ + \Delta t\frac{1}{N}\sum \frac{1}{2}f''(x_{i,t})\sigma^2$$

Fokker-Planck Equation

- Now, we translate the expression, with all our “wishful” $g_t(\cdot)$

$$\int f(x) g_{t+\Delta t}(x)\,\mathrm{d}x - \int f(x) g_t(x)\,\mathrm{d}x$$
$$= \Delta t \int f'(x)\mu(x) g_t(x)\,\mathrm{d}x + \Delta t \int \frac{1}{2}\sigma^2 f''(x) g_t(x)\,\mathrm{d}x$$

Fokker-Planck Equation

$$\int f'(x)\mu(x)g_t(x)\,\mathrm{d}x = f(x)\mu(x)g_t(x)\big|_{\text{boundary}} - \int f(x)\frac{d}{dx}\left(\mu(x)g_t(x)\right)\mathrm{d}x$$

Boundary conditions matter, but we leave it for later, then we have it is zero for now.

$$\int f'(x)\mu(x)g_t(x)\,\mathrm{d}x = -\int f(x)\frac{d}{dx}\left(\mu(x)g_t(x)\right)\mathrm{d}x$$

Fokker-Planck Equation

- Similar application of the integration-by-parts results in

$$\int f''(x) g_t(x) \, \mathrm{d}x = \int f(x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} g_t(x) \, \mathrm{d}x$$

Fokker-Planck Equation

- Collecting everything together, we get

$$\int f(x)\frac{g_{t+\Delta t}(x) - g_t(x)}{\Delta t} = -\int f(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu(x)g(x)\right)\mathrm{d}x + \int f(x)\frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}g_t(x)\,\mathrm{d}x$$

- Note that we never defined what $f(\cdot)$ is.
- This means that $g(\cdot)$ need to satisfy these conditions at all "points"[1]

[1] Refer to the calculus of variations for technical details. Also, this is a solution concept that's relevant for the Fokker-Planck equation in fact.

Fokker-Planck Equation

- Hence, we have

$$\frac{g_{t+\Delta t}(x) - g_t(x)}{\Delta t} = -\frac{\mathrm{d}}{\mathrm{d}x}(\mu(x)g(x)) + \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x)$$

- Hence, in limit

$$\frac{\mathrm{d}g}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}x}(\mu(x)g(x)) + \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x)$$

- This is the Fokker-Planck equation

Fokker-Planck Equation

- Again, what did we gain?
- Went from a population of

$$\mathrm{d}x_{i,t} = \mu(x_{i,t})\,\mathrm{d}t + \sigma\,\mathrm{d}W_{i,t}$$

- to

$$\frac{\mathrm{d}g}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}x}(\mu(x)g(x)) + \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x)$$

Fokker-Planck Equation

- Nicer to approximate numerically[2]

[2] in low dimension

Discretization Methods

- HJB equations are HARD discretization problems.
- FPK equations are nicer, and we have more room for decision making.
- Part of this is because we do not have $\max_a \ldots$ where a^* depends on the value function.
- We have a different problems of
  - positivity
  - preservation of mass
- but they are still easier.

Finite Difference Method

- We can do the same finite-difference methods as in the HJB equation case

$$\frac{\mathrm{d}g}{\mathrm{d}t} = 0 = \frac{\mathrm{d}}{\mathrm{d}x}(-\mu(x)g(x)) + \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x)$$

- i.e.,

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}(-\mu(x)\cdot g(x))\right]_i + \frac{\sigma^2}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x)\right]_i = 0$$

Finite Difference Method

- For example, if we take forward difference, we get

$$\begin{aligned}
&\left[-\frac{\mu(x_{i+1})}{x_{i+1}-x_i}\cdot g(x_{i+1})+\frac{\mu(x_i)}{x_{i+1}-x_i}\cdot g(x_i)\right] \\
&\quad+\frac{\sigma^2}{2}\left[2\frac{g(x_{i-1})}{(x_i-x_{i-1})(x_{i+1}-x_{i-1})}\right. \\
&\quad-2\frac{g(x_i)}{(x_i-x_{i-1})(x_{i+1}-x_i)} \\
&\quad\left.+2\frac{g(x_{i+1})}{(x_{i+1}-x_i)(x_{i+1}-x_{i-1})}\right]=0
\end{aligned}$$

Finite Difference Method

- This expression looks terrible

Finite Difference Method

- This expression looks terrible
- But... everything is linear in $g(x_i)$
- Hence, we end up with the same matrix form as with HJB equation.

$$Ag = 0$$

Finite Difference Method

- For example, the A from above with forward difference is given by

$$
\begin{aligned}
A(i,i-1) &= \frac{\sigma^2}{(x_i - x_{i-1})(x_{i+1} - x_i)} \\
A(i,i) &= \frac{\mu(x_i)}{x_{i+1} - x_i} + \frac{\sigma^2}{(x_i - x_{i-1})(x_{i+1} - x_i)} \\
A(i,i+1) &= -\frac{\mu(x_{i+1})}{x_{i+1} - x_i} + \frac{\sigma^2}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})}
\end{aligned}
$$

- Again, this can be implemented exactly the same as with HJB.

Finite Difference Method

- In fact, if we have use the same grid as that for the HJB equation, A_{FP} is exactly the transform of A_{HJB}.
- My Advisor (Benjamin Moll): "You get distribution for free!"
- This is also good if you use uniformly-spaced grids.
  - Not good for non-uniformly spaced grids, and an adjustment is necessary.[3]

[3] Achdou, Yves, et al. Income and wealth distribution in macroeconomics: A continuous-time approach. No. w23732. National Bureau of Economic Research, 2017.

We Need a Cute Cat Picture about Now...

[3] https://xkcd.com/231/

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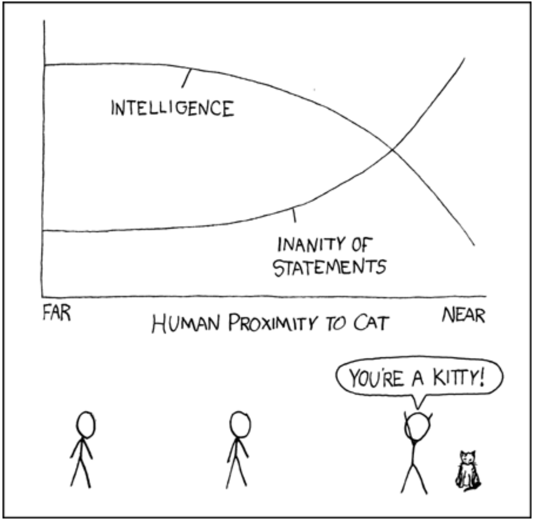

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