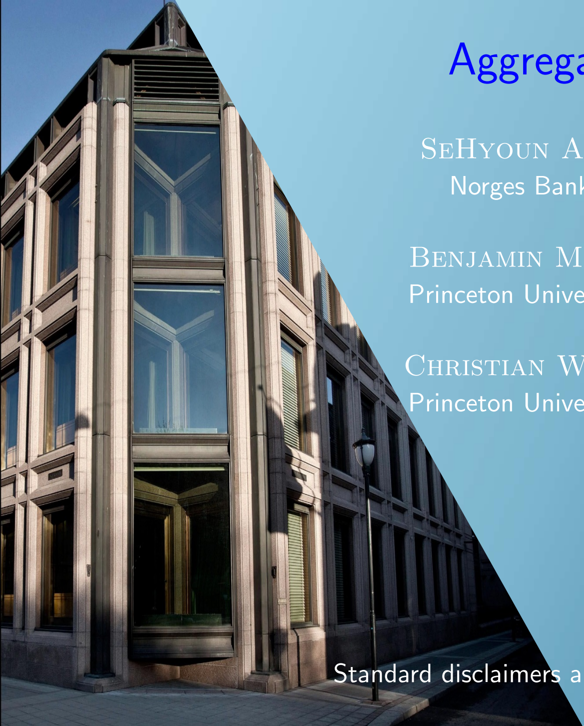

Aggregate Shocks

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2019/11/5

Standard disclaimers apply.

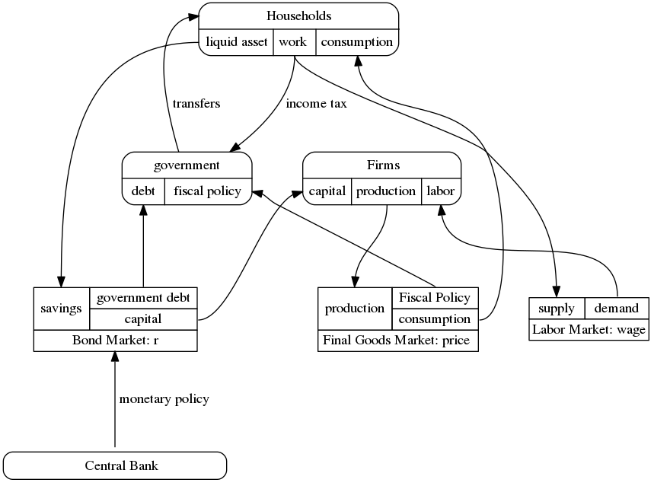
Overview of Our Paper

- Heterogeneous agent models study interaction of macro + inequality
- Not yet part of policymakers' toolbox. Two excuses:
  - Computational difficulties because distribution endogenous
  - Perception that aggregate dynamics similar to representative agent

These excuses less valid than you thought

1. Efficient and easy-to-use computational method
   - Open source `Matlab` toolbox online now
2. Use methodology to illustrate interaction of macro + inequality
   - Match micro behavior $\Longrightarrow$ realistic aggregate C + Y dynamics

Big Picture: Standard DSGE



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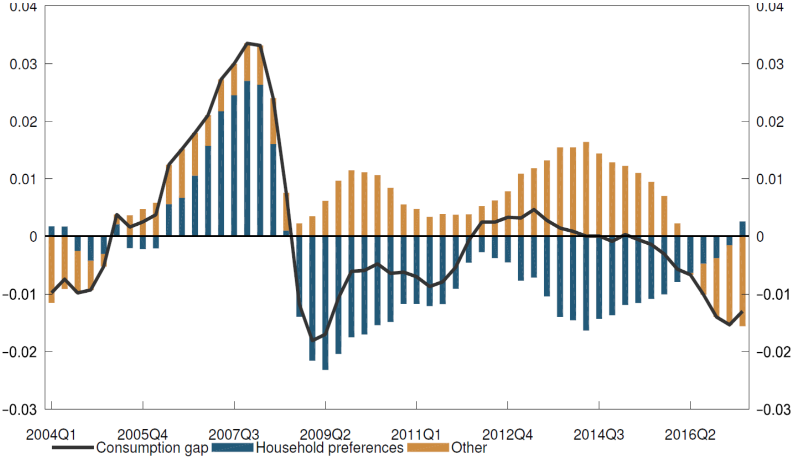
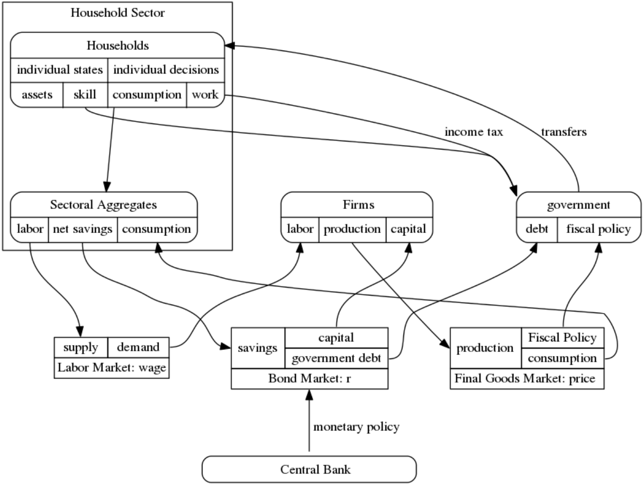
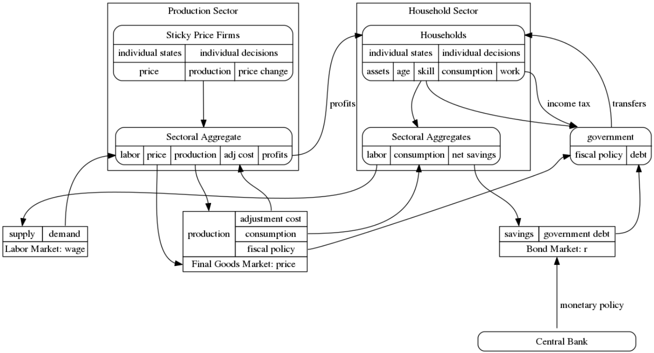


Big Picture: HA-DSGE



Big Picture: HA-DSGE



Plan For Today

1. **Computational Methodology**
   - Simple Krusell-Smith model
   - Linearizing heterogeneous agent models
   - Dimensionality reduction

2. **Applications**
   - Two-asset model
   - Aggregate consumption dynamics
   - Inequality dynamics

Plan For Today

1. **Computational Methodology**
   - Simple Krusell-Smith model
   - Linearizing heterogeneous agent models (Reiter, Campbell, Dotsey-King-Wollman)
   - Dimensionality reduction (model reduction in engineering)

2. **Applications**
   - Two-asset model
   - Aggregate consumption dynamics
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Households

$$\max_{\{c_{jt}\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_{jt}) dt \qquad \text{such that}$$

$$c_{jt} + \dot{a}_{jt} = w_t z_{jt} + r_t a_{jt}$$

$$z_{jt} \in \{z_\ell, z_h\} \text{ Poisson with intensities } \lambda_\ell, \lambda_h$$

$$a_{jt} \geq 0$$

- c_{jt}: consumption
- u: utility function, $u' > 0, u'' < 0$.
- ρ: discount rate
- r_t : interest rate

Production and Market Clearing

- Aggregate production function

$$Y_t = e^{Z_t} K_t^{\alpha} N_t^{1-\alpha} \text{ with } dZ_t = -\nu Z_t + \sigma dW_t$$

- Perfect competition in factor markets

$$w_t = (1-\alpha)\frac{Y_t}{N_t}, \qquad r_t = \alpha \frac{Y_t}{K_t} - \delta$$

- Market clearing

$$K_t = \int a g_t(a,z) da dz,$$
$$N_t = \int z g_t(a,z) da dz \equiv 1$$

Equilibrium

Aggregate state: $(g_t, Z_t) \Rightarrow$ absorb into time subscript t

- Recursive notation w.r.t. individual states only
- $\mathbb{E}_t$ is expectation w.r.t. aggregate states only ▸ fully recursive

Equilibrium

Aggregate state: $(g_t, Z_t) \Rightarrow$ absorb into time subscript t

- Recursive notation w.r.t. individual states only
- $\mathbb{E}_t$ is expectation w.r.t. aggregate states only ▸ fully recursive

$$\rho v_t(a,z) = \max_c \; u(c) + \partial_a v_t(a,z)(w_t z + r_t a - c) + \lambda_z(v_t(a,z') - v_t(a,z)) + \frac{1}{dt}\mathbb{E}_t\left[dv_t(a,z)\right], \tag{HJB}$$

$$\frac{\mathrm{d}g_t(a,z)}{\mathrm{d}t} = -\partial_a[s_t(a,z)g_t(a,z)] - \lambda_z g_t(a,z) + \lambda_{z'} g_t(a,z'), \tag{KF}$$

$$w_t = (1-\alpha)e^{Z_t}K_t^{\alpha} \text{ and } r_t = \alpha e^{Z_t}K_t^{\alpha-1} - \delta, \tag{P}$$

$$K_t = \int a g_t(a,z)\,da\,dz,$$

$$dZ_t = -\nu Z_t dt + \sigma dW_t \tag{Z}$$

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Extending Linearization to Heterogeneous Agent Models

1. Compute non-linear approx. of non-stochastic steady state

2. Compute first-order Taylor expansion around steady state

3. Solve linear stochastic differential equation

Warm Up: Linearizing a Representative Agent Model

- Representative agent RBC model

$$\mathbb{E}_t\left[\mathrm{d}C_t^{-\gamma}\right] = C_t^{-\gamma}\left(\alpha e^{Z_t}K_t^{\alpha-1} - \rho - \delta\right)\mathrm{d}t$$
$$\mathrm{d}K_t = \left(e^{Z_t}K_t^{\alpha} - \delta K_t - C_t\right)\mathrm{d}t$$
$$\mathrm{d}Z_t = -\eta Z_t \mathrm{d}t + \sigma \mathrm{d}W_t$$

- Classification of variables

$$C_t = \text{ control variable}$$
$$K_t = \text{ endogenous state variable}$$
$$Z_t = \text{ exogenous state variable}$$

Warm Up: Linearizing a Representative Agent Model

- Linearized representative agent RBC model

$$\mathbb{E}_t \begin{bmatrix} \mathrm{d}\widehat{C}_t \\ \mathrm{d}\widehat{K}_t \\ \mathrm{d}Z_t \end{bmatrix} = \begin{bmatrix} B_{CC} & B_{CK} & B_{CZ} \\ B_{KC} & B_{KK} & B_{KZ} \\ 0 & 0 & -\eta \end{bmatrix} \begin{bmatrix} \widehat{C}_t \\ \widehat{K}_t \\ Z_t \end{bmatrix} \mathrm{d}t$$

- Classification of variables

$$\begin{aligned} C_t &= \text{ control variable} \\ K_t &= \text{ endogenous state variable} \\ Z_t &= \text{ exogenous state variable} \end{aligned}$$

Extending Linearization to Heterogeneous Agent Models

1. **Compute non-linear approx. of non-stochastic steady state**

2. Compute first-order Taylor expansion around steady state

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Extending Linearization to Heterogeneous Agent Models

1. **Compute non-linear approx. of non-stochastic steady state**
   - Finite difference method from Achdou et al. (2015)
   - Steady state reduces to sparse matrix equations
   - Borrowing constraint absorbed into boundary conditions
2. Compute first-order Taylor expansion around steady state
3. Solve linear stochastic differential equation

Step 1: Compute Non-Stochastic Steady State

$$
\begin{aligned}
\rho v(a,z) = & \max_c \; u(c) + \partial_a v(a,z)(wz + ra - c) \\
& + \lambda_z (v(a,z') - v(a,z))
\end{aligned} \qquad \text{(HJB SS)}
$$

$$
0 = -\partial_a [s(a,z) g(a,z)] - \lambda_z g(a,z) + \lambda_{z'} g(a,z') \qquad \text{(KF SS)}
$$

$$
\begin{aligned}
w &= (1-\alpha)K^{\alpha}, \quad r = \alpha K^{\alpha-1} - \delta, \\
K &= \int a g(a,z) da dz
\end{aligned} \qquad \text{(P SS)}
$$

Step 1: Compute Non-Stochastic Steady State

$$\begin{aligned}\rho v_{i,j} =& u(c_{i,j}) + \partial_a v_{i,j}(wz_j + ra_i - c_{i,j}) \\ &+ \lambda_j(v_{i,-j} - v_{i,j}), \text{ with } c_{i,j} = u'^{-1}(\partial_a v_{i,j})\end{aligned} \tag{HJB SS}$$

$$0 = -\partial_a[s(a,z)g(a,z)] - \lambda_z g(a,z) + \lambda_{z'} g(a,z') \tag{KF SS}$$

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Step 1: Compute Non-Stochastic Steady State

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}; \mathbf{p})\,\mathbf{v} \tag{HJB SS}$$

$$0 = -\partial_a [s(a,z) g(a,z)] - \lambda_z g(a,z) + \lambda_{z'} g(a,z') \tag{KF SS}$$

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$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v};\mathbf{p})\,\mathbf{v} \qquad \text{(HJB SS)}$$

$$\mathbf{0} = \mathbf{A}(\mathbf{v};\mathbf{p})^{\mathrm{T}}\,\mathbf{g} \qquad \text{(KF SS)}$$

$$\mathbf{p} = \mathbf{F}(\mathbf{g}) \qquad \text{(P SS)}$$

Linearizing Continuous Time Het Agent Models

1. Compute non-linear approximation to non-stochastic steady state
   - Finite difference method from Achdou et al. (2015)
   - Steady state reduces to sparse matrix equations
   - Borrowing constraint absorbed into boundary conditions
2. **Compute first-order Taylor expansion around steady state**
3. Solve linear stochastic differential equation

Linearizing Continuous Time Het Agent Models

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2. **Compute first-order Taylor expansion around steady state**
   - Automatic differentiation: exact numerical derivatives
   - Efficient `Matlab` implementation for sparse systems
3. Solve linear stochastic differential equation

Step 2: Linearize Discretized System

- Discretized system with aggregate shocks

$$\rho \mathbf{v}_t = \mathbf{u}(\mathbf{v}_t) + \mathbf{A}(\mathbf{v}_t;\mathbf{p}_t)\,\mathbf{v}_t + \frac{1}{dt}\mathbb{E}_t[d\mathbf{v}_t]$$

$$\frac{d\mathbf{g}_t}{dt} = \mathbf{A}(\mathbf{v}_t;\mathbf{p}_t)^{\mathrm{T}}\,\mathbf{g}_t$$

$$\mathbf{p}_t = \mathbf{F}(\mathbf{g_t};Z_t)$$

$$dZ_t = -\nu Z_t dt + \sigma dW_t$$

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- Discretized system with aggregate shocks

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\mathbf{p}_t &= \mathbf{F}\left(\mathbf{g_t}; Z_t\right) \\
dZ_t &= -\nu Z_t dt + \sigma dW_t
\end{aligned}
$$

- Write in general form

$$
\mathbb{E}_t \begin{bmatrix} d\mathbf{v}_t \\ d\mathbf{g}_t \\ \mathbf{0} \\ dZ_t \end{bmatrix} = f(\mathbf{v}_t, \mathbf{g}_t, \mathbf{p}_t, Z_t)dt, \qquad \begin{bmatrix} \mathbf{v}_t \\ \mathbf{g}_t \\ \mathbf{p}_t \\ Z_t \end{bmatrix} = \begin{bmatrix} \text{control} \\ \text{endog state} \\ \text{prices} \\ \text{exog state} \end{bmatrix}
$$

Step 2: Linearize Discretized System

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\mathbf{p}_t &= \mathbf{F}(\mathbf{g}_\mathbf{t};Z_t) \\
dZ_t &= -\nu Z_t dt + \sigma dW_t
\end{aligned}$$

- Linearize using automatic differentiation (code: `@myAD`)

$$\mathbb{E}_t \begin{bmatrix} d\widehat{\mathbf{v}}_t \\ d\widehat{\mathbf{g}}_t \\ \mathbf{0} \\ dZ_t \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{vv} & \mathbf{0} & \mathbf{B}_{vp} & \mathbf{0} \\ \mathbf{B}_{gv} & \mathbf{B}_{gg} & \mathbf{B}_{gp} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{pg} & -\mathbf{I} & \mathbf{B}_{pZ} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{v}}_t \\ \widehat{\mathbf{g}}_t \\ \widehat{\mathbf{p}}_t \\ Z_t \end{bmatrix} dt$$

Linearizing Continuous Time Het Agent Models

1. Compute non-linear approximation to non-stochastic steady state
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3. **Solve linear stochastic differential equation**
    - Moderately-sized systems $\Longrightarrow$ standard methods OK

Step 3: Solve Linear System

- Diagonalize + hope that number of stable eigenvalues = number of state variables

- Set control variables $\perp$ unstable eigenvectors $\implies$ policy function

$$\widehat{\mathbf{v}}_t = \mathbf{D}_g \widehat{\mathbf{g}}_t + \mathbf{D}_Z \widehat{Z}_t$$

- Feasible for $N \leq 5000$ or so

Linearization is Fast and Accurate

- Calibration: JEDC (2010) comparison project on Krusell-Smith
- Size: 100 asset grid points $\implies$ total system ≈ 400

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- Size: 100 asset grid points $\Longrightarrow$ total system ≈ 400
- Speed: $\approx$ `0.25 seconds`
  - JEDC (2010) project: $\approx$ `7 minutes` up to $\approx$ `46 hours`
- Accuracy: Max difference in K_t from simulations using individual policies vs. aggregate law of motion

Agg Shock σ	0.01%	0.1%	0.7%	1%	5%
DH Error Stat	0.000%	0.002%	0.053%	0.135%	3.347%

  - JEDC (2010) project: most accurate alternative $\approx 0.16\%$

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Model-Free Reduction Method

$$\mathbb{E}_t \begin{bmatrix} d\widehat{\mathbf{v}}_t \\ d\widehat{\mathbf{g}}_t \\ \mathbf{0} \\ dZ_t \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{vv} & \mathbf{0} & \mathbf{B}_{vp} & \mathbf{0} \\ \mathbf{B}_{gv} & \mathbf{B}_{gg} & \mathbf{B}_{gp} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{pg} & -\mathbf{I} & \mathbf{B}_{pZ} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{v}}_t \\ \widehat{\mathbf{g}}_t \\ \widehat{\mathbf{p}}_t \\ Z_t \end{bmatrix} dt$$

- Dimensionality: 2 income types $\times$ M wealth grid points $\implies$ both $\mathbf{v}_t$ and $\mathbf{g}_t$ are $N(=2M) \times 1$ vectors

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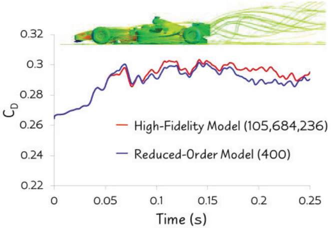
  $\implies$ both $\mathbf{v}_t$ and $\mathbf{g}_t$ are $N(= 2M) \times 1$ vectors

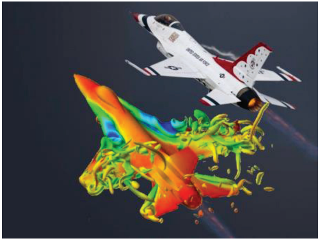
1. Value function: reduce using quadratic splines
   - Will not discuss today

2. Distribution: reduce using model reduction tools
   - Explain intuition in special cases
   - Paper has detailed proofs

Distribution Reduction by Projection

Or, what race cars and fighter jets can teach us about distributional dynamics





include

Based on Stanford Computational and Mathematical Engineering (CME) 345 "Model Reduction"

`https://web.stanford.edu/group/frg/course_work/CME345.html`

Distribution Reduction by Projection

- Key insight: households only need to forecast prices
  - Krusell-Smith: guess moments to approx distribution, check they forecast prices
  - Our approach: have computer choose "moments", guarantees accuracy

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- Distribution exactly reduces if there exists as basis $\mathbf{X} = [\mathbf{x}_1, ..., \mathbf{x}_k]$ such that

$$\mathbf{g}_t = \gamma_{1t}\mathbf{x}_1 + \gamma_{2t}\mathbf{x}_2 + ... + \gamma_{kt}\mathbf{x}_k \equiv \mathbf{X}\gamma_t$$

  - N-dimensional $\mathbf{g}_t$ approximated with $k << N$-dimensional γ_t
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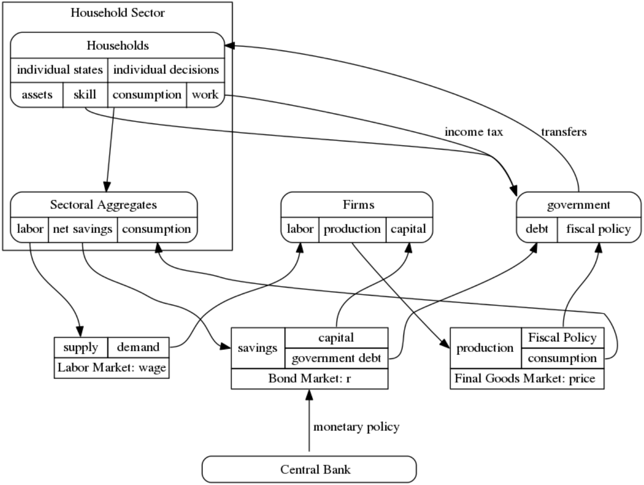
$\Longrightarrow$ Goal: Choose $\mathbf{X}$ to "approximate" IRFs of $\mathbf{p}_t$ with small k

Big Picture: HA-DSGE



A Special Case: Exogenous Decision Rules

- Suppose given $\mathbf{D}_{vg}$ and $\mathbf{D}_{vZ}$ in $\mathbf{v}_t = \mathbf{D}_{vg}\mathbf{g}_t + \mathbf{D}_{vZ}Z_t$

$$\frac{\mathrm{d}\mathbf{g}_t}{\mathrm{d}t} = \mathbf{C}_{gg}\mathbf{g}_t + \mathbf{C}_{gZ}Z_t$$

$$\mathbf{p}_t = \mathbf{B}_{pg}\mathbf{g}_t + \mathbf{B}_{pZ}Z_t$$

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- Protoypical problem in `model reduction literature`
  - Maps low-dimensional inputs (Z_t) into low-dimensional outputs $(\mathbf{p}_t)$
  - High-dimensional intermediating variable $(\mathbf{g}_t)$

- To reduce distribution, need to
  1. Find a good basis $\mathbf{X}$
  2. Given basis $\mathbf{X}$, estimate coefficients γ_t

Plan Of Attack

1. **Exogenous decision rules**: adapt existing results

   - Start in deterministic model ($Z_t = 0$ for all t)

   $$\frac{\mathrm{d}\mathbf{g}_t}{\mathrm{d}t} = \mathbf{C}_{gg}\mathbf{g}_t$$
   $$\mathbf{p}_t = \mathbf{B}_{pg}\mathbf{g}_t$$

   given initial $\mathbf{g}_0$

   - Move to stochastic model

2. **Endogenous decision rules**

Plan Of Attack

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$$\frac{\mathrm{d}\mathbf{g}_t}{\mathrm{d}t} = \mathbf{C}_{gg}\mathbf{g}_t$$
$$p_t = \mathbf{b}_{pg}\mathbf{g}_t \qquad \text{(a scalar)}$$

     given initial $\mathbf{g}_0$

   - Move to stochastic model

2. **Endogenous decision rules**

Estimating Coefficients Given Basis $\mathbf{X}$

- Can write $\mathbf{g}_t \approx \mathbf{X}\gamma_t$ as a linear regression

$$\mathbf{g}_t = \mathbf{X}\gamma_t + \varepsilon_t, \quad \varepsilon_t \in \mathbb{R}^N = \text{ residual}$$

  - $\mathbf{g}_t$ = dependent variable
  - $\mathbf{X} = [\mathbf{x}_1, ..., \mathbf{x}_k]$ contains k independent variables
  - γ_t = coefficients to be estimated

- Estimate γ_t using the orthogonality condition $\mathbf{X}^{\mathrm{T}}\varepsilon_t = 0$

$$\gamma_t = \underbrace{(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}}_{=\mathbf{I}}\mathbf{X}^{\mathrm{T}}\mathbf{g}_t$$

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  - γ_t = coefficients to be estimated

- Estimate γ_t using the orthogonality condition $\mathbf{X}^{\mathrm{T}}\varepsilon_t = 0$

$$\gamma_t = \underbrace{(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}}_{=\mathbf{I}}\mathbf{X}^{\mathrm{T}}\mathbf{g}_t$$

- Reduced system is

$$\widetilde{p}_t = \mathbf{b}_{pg}\mathbf{X}\gamma_t$$

$$\frac{d\gamma_t}{dt} = \mathbf{X}^{\mathrm{T}}\mathbf{C}_{gg}\mathbf{X}\gamma_t$$

How To Choose Basis X?

- Choose basis $\mathbf{X}$ to match transition path of p_t
  $\Longrightarrow$ match k-order Taylor expansion of p_t using only γ_t

How To Choose Basis $\mathbf{X}$?

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- Unreduced model:

$$
\begin{aligned}
p_t &= \mathbf{b}_{pg}\mathbf{g}_t \\
\frac{d\mathbf{g}_t}{dt} &= \mathbf{C}_{gg}\mathbf{g}_t
\end{aligned}
$$

- Reduced model:

$$
\begin{aligned}
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\end{aligned}
$$

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- Unreduced model:

$$p_t = \mathbf{b}_{pg} e^{\mathbf{C}_{gg} t} \mathbf{g}_0$$

- Reduced model:

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  $\Longrightarrow$ match k-order Taylor expansion of p_t using only γ_t

- Unreduced model:

$$p_t \approx \mathbf{b}_{pg} \left[\mathbf{I} + \mathbf{C}_{gg} t + \frac{1}{2} \mathbf{C}_{gg}^2 + ... \right] \mathbf{g}_0$$

- Reduced model:

$$\widetilde{p}_t \approx \mathbf{b}_{pg} \mathbf{X} \left[\mathbf{I} + (\mathbf{X}^{\mathrm{T}} \mathbf{C}_{gg} \mathbf{X}) t + \frac{1}{2} (\mathbf{X}^{\mathrm{T}} \mathbf{C}_{gg} \mathbf{X})^2 + ... \right] \gamma_0$$

How To Choose Basis $\mathbf{X}$?

- Choose basis $\mathbf{X}$ to match transition path of p_t
  $\Longrightarrow$ match k-order Taylor expansion of p_t using only γ_t

- **Claim**: if $\mathbf{X}$ spans $\mathcal{O}(\mathbf{b}_{pg}, \mathbf{C}_{gg})^{\mathrm{T}}$, then path of reduced $\widetilde{p}_t$ matches path unreduced of p_t up to order k

$$\mathcal{O}(\mathbf{b}_{pg}, \mathbf{C}_{gg}) := \begin{bmatrix} \mathbf{b}_{pg} \\ \mathbf{b}_{pg}\mathbf{C}_{gg} \\ \mathbf{b}_{pg}\mathbf{C}_{gg}^2 \\ \vdots \\ \mathbf{b}_{pg}\mathbf{C}_{gg}^{k-1} \end{bmatrix}$$

- Why $\mathcal{O}(\mathbf{b}_{pg}, \mathbf{C}_{gg})$?
  $p_t \approx \left[1, t, \frac{1}{2}t^2, ..., \frac{1}{(k-1)!}t^{k-1}\right] \mathcal{O}(\mathbf{b}_{pg}, \mathbf{C}_{gg})\mathbf{g}_0$

How To Choose Basis X?

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How To Choose Basis X In Stochastic Model?

- Choose basis $\mathbf{X}$ to match impulse response of p_t to Z_t shock
- **Claim**: If $\mathbf{X}$ spans order k observability matrix $\mathcal{O}(\mathbf{b}_{pg}, \mathbf{C}_{gg})^{\mathrm{T}}$, then IRF of reduced $\widetilde{p}_t$ matches IRF of unreduced p_t up to order k

How To Choose Basis X In Stochastic Model?

- Choose basis $\mathbf{X}$ to match impulse response of p_t to Z_t shock
- **Claim**: If $\mathbf{X}$ spans order k observability matrix $\mathcal{O}(\mathbf{b}_{pg}, \mathbf{C}_{gg})^{\mathrm{T}}$, then IRF of reduced $\widetilde{p}_t$ matches IRF of unreduced p_t up to order k
- **Intuition**: Impulse response combines
  1. Impact effect: do not reduce $Z_t \implies$ match exactly
  2. Transition to steady state: role of $\mathcal{O}(\mathbf{b}_{pg}, \mathbf{C}_{gg})$

Extending To Endogenous Decision Rules

- Model reduction literature relies on reduction not affecting dynamics

$$\begin{aligned} \mathbf{C}_{gg} &= \mathbf{B}_{gg} + \mathbf{B}_{gp}\mathbf{B}_{pg} + \mathbf{B}_{gv}\mathbf{D}_{vg} \\ \mathbf{C}_{gZ} &= \mathbf{B}_{gp}\mathbf{B}_{pZ} + \mathbf{B}_{gv}\mathbf{D}_{vZ} \end{aligned}$$

  - Violated with endogenous decision rules

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$$\mathbf{C}_{gZ} = \mathbf{B}_{gp}\mathbf{B}_{pZ} + \mathbf{B}_{gv}\mathbf{D}_{vZ}$$

  - Violated with endogenous decision rules

- But literature about efficiently approximating the distribution
  - Can inefficiently improve approximation by adding independent basis vectors

- **Solution**: set $\mathbf{X}$ to span $\mathcal{O}(\mathbf{b}_{pg}, \mathbf{C}_{gg})^{\mathrm{T}}$ assuming $\mathbf{D}_{vg} = \mathbf{D}_{vZ} = 0$

- If implied dynamics are inaccurate, then iterate

Internal Consistency

- Key question: when is approximation accurate? I.e., how to choose k?

Internal Consistency

- Key question: when is approximation accurate? I.e., how to choose k?

- **Answer 1**: increase k until IRFs converge

- **Answer 2**: internal consistency check
  1. Compute decisions from reduced model $\widetilde{\mathbf{v}}_t = \mathbf{D}_{v\gamma}\gamma_t + \mathbf{D}_{vZ}Z_t$
  2. Simulate nonlinear dynamics of full distribution
  $$\mathbf{p}_t^* = \mathbf{F}(\mathbf{g}_t^*; Z_t)$$
  $$\frac{d\mathbf{g}_t^*}{dt} = \mathbf{A}(\widetilde{\mathbf{v}}_t, \mathbf{p}_t^*)\mathbf{g}_t^*$$
  3. Compare to dynamics implied by reduced system $\widetilde{\mathbf{p}}_t$
  $$\epsilon = \max_i \max_{t\geq 0} |\log \widetilde{p}_{it} - \log p_{it}^*|$$

The Reduced Linear System

- Summarizing, we approximate

$$\widehat{\mathbf{v}}_t \approx \mathbf{Z}\eta_t,$$
$$\widehat{\mathbf{g}}_t \approx \mathbf{X}\gamma_t,$$

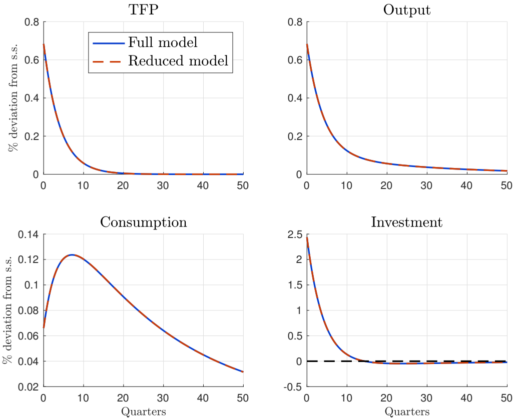
  where η_t is $k_v \times 1$, γ_t is $k_g \times 1$ with $k_v, k_g << N$

- Sufficient to keep track of these low-dimensional vectors:

$$\mathbb{E}_t \begin{bmatrix} d\eta_t \\ d\gamma_t \\ dZ_t \end{bmatrix} = \begin{bmatrix} \mathbf{Z}'\mathbf{B}_{vv}\mathbf{Z} & \mathbf{Z}'\mathbf{B}_{vp}\mathbf{B}_{pg}\mathbf{X} & \mathbf{Z}'\mathbf{B}_{vp}\mathbf{B}_{pZ} \\ \mathbf{X}'\mathbf{B}_{gv}\mathbf{Z} & \mathbf{X}'(\mathbf{B}_{gg} + \mathbf{B}_{gp}\mathbf{B}_{pg})\mathbf{X} & \mathbf{X}'\mathbf{B}_{gp}\mathbf{B}_{pZ} \\ \mathbf{0} & \mathbf{0} & -\nu \end{bmatrix} \begin{bmatrix} \eta_t \\ \gamma_t \\ Z_t \end{bmatrix} d$$

- Then proceed as before

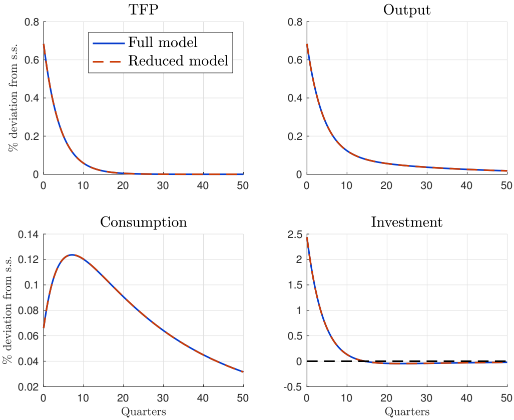
Approximate Aggregation in KS Model



- Comparison of full distribution vs. $k = 1$ approximation

  $\implies$ recovers Krusell & Smith's "approximate aggregation"

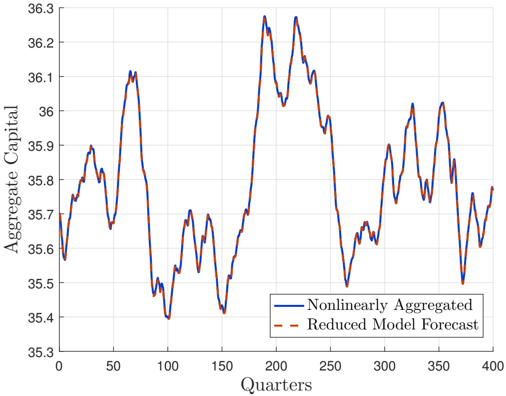
Approximate Aggregation in KS Model



- Large-scale models in applications require $k = 300$

  $\Longrightarrow$ no approximate aggregation

Internal Consistency



- Maximum deviation: 0.065%
- Maximum deviation in unreduced model: 0.049%

Model Reduction Speeds Up Solution

	w/o Reduction	**w/ Reduction**
Steady State	0.082 sec	0.082 sec
Linearize	0.021 sec	0.021 sec
Reduction	$\times$	0.007 sec
Solve	0.14 sec	0.002 sec
Total	0.243 sec	0.112 sec

Plan For Today

1. **Computational Methodology**
   - Simple Krusell-Smith model
   - Linearizing heterogeneous agent models
   - Dimensionality reduction
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Households

$$\max_{\{c_{jt}\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-(\rho+\zeta)t} u(c_{jt}) dt \qquad \text{such that}$$

$$c_{jt} + \dot{b}_{jt} + d_{jt} + \chi(d_{jt}, a_{jt}) = r_t^b(b_{jt}) b_{jt} + w_t z_{jt} - T(w_t z_{jt})$$

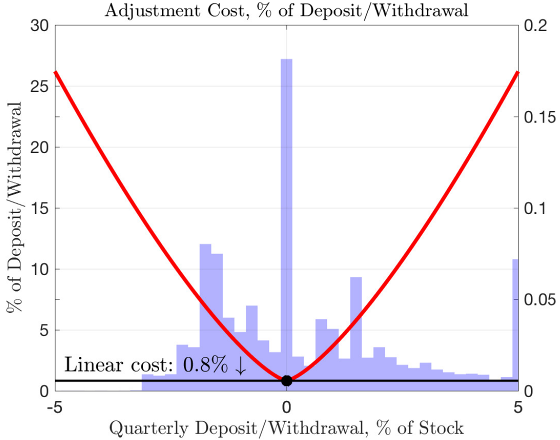
$$\dot{a}_{jt} = r_t^a a_{jt} + d_{jt}$$

$$z_{jt} \in \{z_1, ..., z_{N_z}\} \text{ Poisson with intensities } \lambda_{zz'}$$

$$b_{jt} \geq -\underline{B} \times Z_t \text{ and } a_{jt} \geq 0$$

- b_{jt}: liquid assets
- a_{jt}: illiquid assets
- d_{jt}: illiquid deposits ($\gtrless 0$)
- $\chi(d_{jt}, a_{jt})$: transaction cost function
- $r_t^b(b_{jt}) = r_t^b$ if $b_{jt} \geq 0$, $= r_t^b + \kappa$ if $b_{jt} < 0$

Kinked adjustment cost function $\chi(d, a)$



$$\chi(d_{jt}, a_{jt}) = \chi_0 |d_{jt}| + \chi_1 \left| \frac{d_{jt}}{a_{jt}} \right|^{\chi_2} a_{jt}$$

Production and Market Clearing

- Aggregate production function with growth rate shocks

$$
\begin{aligned}
Y_t &= K_t^{\alpha} \left(Q_t N_t\right)^{1-\alpha} \\
d \log Q_t &= Z_t dt \\
dZ_t &= -\nu Z_t dt + \sigma dW_t
\end{aligned}
$$

- Perfect competition in factor markets

$$
w_t = (1-\alpha)\frac{Y_t}{N_t}, \qquad r_t^a = \alpha \frac{Y_t}{K_t} - \delta
$$

- Market clearing
  - Illiquid assets: $K_t = \int a dG_t(a,b,z)$
  - Liquid assets: $B = \int b dG_t(a,b,z)$
  - Labor market: $N_t = \int z dG_t(a,b,z) \equiv 1$

Parameterization

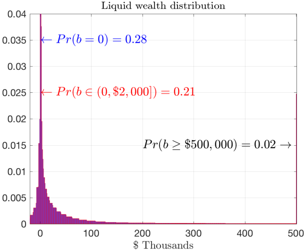
1. Distribution of income and wealth in micro data
   - Exogenously fix subset of parameters to standard values
   - Estimate labor productivity shocks from SSA data ▸ Details
   - Choose transaction costs + discount rate to match wealth distribution

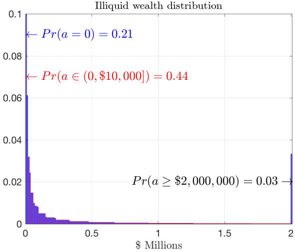
2. Dynamics of income in macro data

Statistic	Data	Model
$\sigma\left(\Delta \log Y_t\right)$	0.89%	0.88%
Corr$(\Delta \log Y_t, \Delta \log Y_{t-1})$	0.37	0.36

$d \log Q_t = Z_t dt$, with $dZ_t = -\nu Z_t dt + \sigma dW_t$

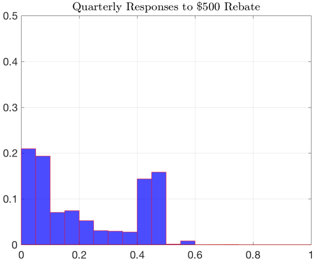
Model matches key feature of U.S. wealth distribution

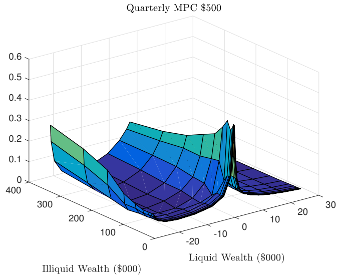




	Data	Model
Mean illiquid assets (rel to GDP)	3.000	3.000
Mean liquid assets (rel to GDP)	0.375	0.375
Poor hand-to-mouth	10.0%	10.5%
Wealthy hand-to-mouth	20.0%	17.2%
Borrowers	15.0%	13.5%

Model generates high and heterogeneous MPCs





- Average quarterly MPC out of a $500 windfall: 23%

Parameterization

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2. Dynamics of aggregate income in macro data

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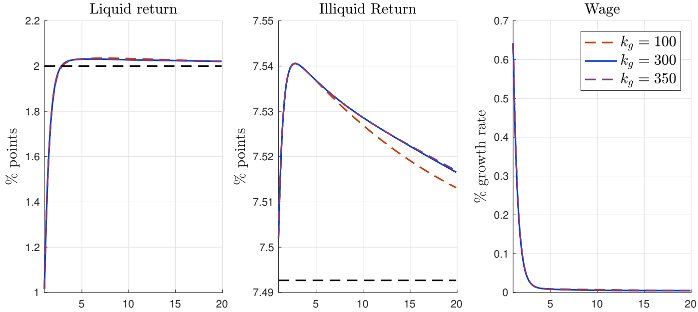
"Approximate Aggregation" Breaks Down



Performance of the Method, Size $\approx 132,000$

	$k_g = 300$	$k_g = 150$
Steady State	47.00 sec	47.00 sec
Derivatives	21.91 sec	21.91 sec
Dim reduction	258.80 sec	79.90 sec
Linear system	17.14 sec	12.66 sec
Simulate IRF	3.76 sec	2.12 sec
Total	**348.61 sec**	**171.58 sec**

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   - Inequality dynamics

Application 1: Inequality Matters for Agg C + Y Dynamics

- Campbell-Mankiw Macro Annual '89: how match C + Y dynamics?

	Data	**Models**	
		Rep agent	Two-Asset
Sensitivity to Income			
IV($\Delta \log C_t$ on $\Delta \log Y_t$ using $\Delta \log Y_{t-1}$)	0.503	0.247	0.656
Smoothness			
$\frac{\sigma(\Delta \log C_t)}{\sigma(\Delta \log Y_t)}$	0.518	0.709	0.514

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Application 2: Agg Shocks Matter for Inequality Dynamics

- With Cobb-Douglas production, labor income inequality exogenous

$$\text{labor income } = w_t \times z_{jt}$$

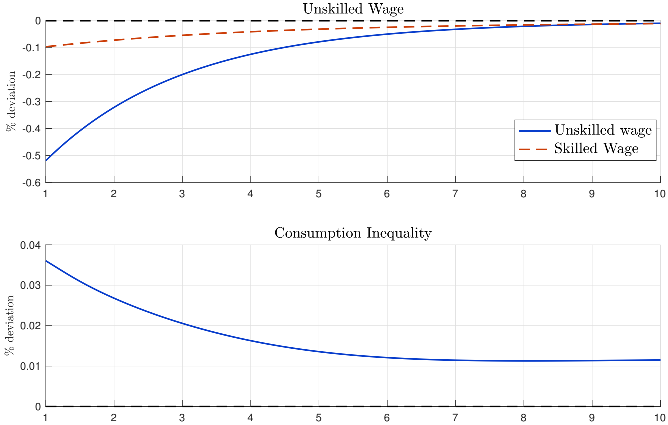
- Modify production function to generate endogenous inequality

$$Y_t = \left[\mu(Z_t^U N_t^U)^\sigma + (1-\mu)\left(\lambda K_t^\rho + (1-\lambda)(N_t^S)^\rho\right)^{\frac{\sigma}{\rho}}\right]^{\frac{1}{\sigma}}$$

  - N_t^U: unskilled labor w/ low persistent productivity z_{jt}
  - N_t^S: skilled labor w/ high persistent productivity z_{jt}
  - Z_t^U: unskilled-specific productivity shock

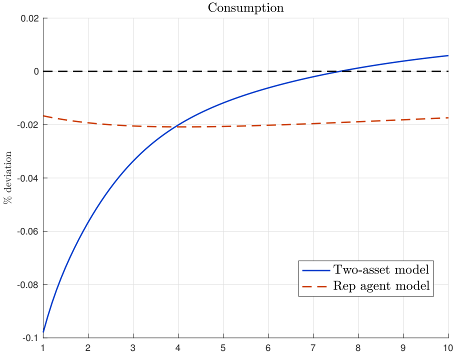
- Calibrate σ and ρ to generate capital-skill complementarity

Unskilled-Specific Shock Increases Inequality...



- Fluctuations in income inequality ≈ aggregate income

... And Generates Sharp Consumption Bust



- Many low-skill households hand-to-mouth

  $\Longrightarrow$ larger consumption drop than in rep agent model

Macro With Inequality: No More Excuses!

1. Efficient and easy-to-use computational method
   - Open source `Matlab` toolbox online now

2. Use methodology to illustrate interaction of macro + inequality
   - Match micro behavior $\implies$ realistic aggregate C + Y dynamics
   - Aggregate shocks generate inequality dynamics

- Estimating models w/ micro data on distributions within reach

Instead: Fully Recursive Notation ▸ Back

$$w(g,Z) = (1-\alpha)e^Z K(g)^\alpha, \quad r(g,Z) = \alpha e^Z K(g)^{\alpha-1} - \delta \qquad \text{(P)}$$

$$K(g) = \int a g(a,z) da dz \qquad \text{(K)}$$

$$\begin{aligned}\rho V(a,z,g,Z) = \max_c \; & u(c) + \partial_a V(a,z,g,Z)[w(g,Z)z + r(g,Z)a - c] \\ & + \lambda_z [V(a,z',g,Z) - V(a,z,g,Z)] \\ & + \partial_Z V(a,z,g,Z)(-\nu Z) + \frac{1}{2}\partial_{ZZ} V(a,z,g,Z)\sigma^2 \\ & + \int \frac{\delta V(a,z,g,Z)}{\delta g(a,z)} T[g,Z](a,z) da dz\end{aligned}$$

(∞d HJB)

$$T[g,Z](a,z) = -\partial_a[s(a,z,g,Z)g(a,z)] - \lambda_z g(a,z) + \lambda_{z'} g(a,z')$$

(KF operator)

$$s(a,z,g,Z) = w(g,Z)z + r(g,Z)a - c^*(a,z,g,Z)$$

- $\delta V/\delta g(a,z)$: functional derivative of V wrt g at point (a,z)

Labor Productivity Shocks ▸ Back

$$\log z_{jt} = z_{1,jt} + z_{2,jt}$$

$$dz_{i,jt} = -\beta_i z_{i,jt} dt + \varepsilon_{i,jt} dN_{i,jt}, \text{ where } \varepsilon \sim N(0, \sigma_i^2) \text{ for } i = 1, 2$$

Moment	Data	Model Estimated	Model Discretized
Variance: annual log earns	0.70	0.70	0.74
Variance: 1yr change	0.23	0.23	0.21
Variance: 5yr change	0.46	0.46	0.49
Kurtosis: 1yr change	17.8	16.5	15.5
Kurtosis: 5yr change	11.6	12.1	13.2
Frac 1yr change $< 10\%$	0.54	0.56	0.63
Frac 1yr change $< 20\%$	0.71	0.67	0.71
Frac 1yr change $< 50\%$	0.86	0.85	0.83

Labor Productivity Shocks ▸ Back

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Parameter		Component $j=1$	Component $j=2$
Arrival rate	λ_j	0.080	0.007
Mean reversion	β_j	0.761	0.009
St. Deviation of innovations	σ_j	1.74	1.53