SEQUENCING PROBLEMS WITH PARTICIPATION CONSTRAINTS AND INCENTIVES

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ABSTRACT. We study how participation constraint impacts sequencing problems with monetary transfers where the sequencing rule is outcome efficient and the mechanism is strategyproof. We first identify a property which we call the "interval property" that is both necessary and sufficient to obtain mechasnisms satisfying outcome efficiency, strategyproofness and participation constraints. Given the interval property, we also identify the complete set of all mechanims satisfying outcome efficiency, strategyproofness and participation constraints. We then apply our results to sequencing problems with and without initial orders.

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1. Introduction

The objective of this paper is to analyze the impact of participation constraints in a mechanism design problem. Our study revolves around sequencing problems with a finite set of agents. Each agent has one job to process using a facility that can only serve one agent's requirement at a time. It is further assumed that no job can be interrupted once it starts processing. A job is characterized by its processing time and an agent's waiting cost. The latter represents the dis-utility of waiting (per unit of time) and agents are assumed to have quasi-linear preferences. There is a well established literature in this direction.[1] A special case of sequencing problems where the processing times of the agents are identical is called queueing problems. Queueing problems have also been analyzed extensively from both normative and strategic viewpoints.[2]

A well-known and well studied concept is the outcome efficient sequencing rule that minimizes the aggregate job completion cost of the agents. As pointed out by Smith [32], outcome efficiency requires that the jobs of the agents should be processed in the non-increasing order of their urgency index. The urgency index of any agent is the ratio of his waiting cost and his processing time. Given, we want to implement outcome efficient sequencing rule when waiting cost of the agents are private information, we have a mechanism design set up under incomplete information. It is also well-known that, as long as preferences are 'smoothly connected' (see Holmström [21]), outcome efficient rules can be implemented in dominant strategies (or equivalently we can achieve outcome efficiency and strategyproofness) if and only if the mechanism is a Vickrey-Clarke-Groves (VCG) mechanism (see Clarke [4], Groves [17] and Vickrey [35]). For the sequencing problem, such mechanism design set up under incomplete information was analyzed by Dolan [15], Mitra [25] and Suijs [34].

In this paper we assume that the processing time of all agents are publicly known while the waiting costs are private information and we want mechanisms to implement the outcome efficient rule in dominant strategies with the added restriction of participation constraints. In our first result, we identify the "interval property" (which is an agent specific restriction in

[1] See De [11], [12], De and Mitra [13], [14], Dolan [15], Duives, Heydenreich, Mishra, Muller and Uetz [16], Hain and Mitra [19], Mitra [25], Moulin [28] and Suijs [34].

[2] See Chun [2], [3], Chun, Mitra and Mutuswami [5], [6], [7], [8], Hashimoto and Saitoh [20], Kayi and Ramaekars [22], Maniquet [23], Mitra [24], [26], Mitra and Mutuswami [27] and Mukherjee [31].

terms of processing times of all agents) that is both necessary and sufficient to find mechanisms satisfying outcome efficiency, strategyproofness and participation constraints. Clearly, given the interval property and given Holmström's [21] result on VCG mechanisms, any mechanism that satisfies outcome efficiency, strategyproofness and participation constraints must be a VCG mechanism satisfying participation constraints. Given interval property, our second result identifies the complete set of VCG mechanisms satisfying participation constraints.

We then apply these two general results in two different frameworks. The first framework is sequencing problems with initial order where there is a preexisting order on the agents (may be based on their order of arrivals or based on some factors beyond the control of the mechanism designer). From the cooperative game perspective, this type of sequencing problem with initial order was analyzed for sequencing games by Curiel, Pederzoli, Tijs [10] and, from the mechanism design perspective, for the queueing problems this problem was addressed by by Chun, Mitra and Mutuswami [7] and by Gershkov and Schweinzer [18]. The second framework is sequencing problems without initial orders where there is no preexisting order on the agents. To the best of our knowledge such a problem is new in the sequencing problem but has been analyzed by Chun and Yengin [9], Kayi and Ramaekers [22] and Mitra [26] in the queueing context. For sequencing problems with and without initial order, we also try to identify mechanisms that in addition satisfy either feasibility or its stronger version called budget balance. It is well-known that feasibility of a mechanism requires that the sum of transfers across all agents is non-positive and budget balancedness requires that the sum of transfers across all agents is zero.

For sequencing problems with a given initial order, it is natural to conceive participation constraint of any agent as the agent's total cost under the given initial order. Then achieving outcome efficiency and eliciting private information boils down to reordering this existing initial order to the outcome efficient order by using VCG transfers. In this context it is easy to show that any sequencing problems with a given initial order satisfies the agent specific restriction in terms of processing times of the agents and hence, given our general results, we can easily identify the complete set of all VCG mechanims satisfying participation constraints. More importantly, we can also show that there is no feasible (and hence no budget balanced) mechanism in this class. For the queueing problem this issue was addressed by Chun, Mitra and Mutuswami [7] and our result generalizes it to the sequencing problems.

For sequencing problems with no initial order, participation constraint of an agent is a 'reasonable' upper bound on the cost that can be guaranteed to each agent if the agent participates in the mechanism. In this set up we apply two notions of participation constraints-the identical costs lower bound (ICLB) and the expected cost lower bound (ECLB). ICLB is a well-known notion of fairness used in many context.[3] ICLB requires that each agent receives at least the utility he could expect under the egalitarian solution if all agents were like him in a reference economy. The reference economy for any agent i requires that all other agents have the same waiting cost and processing time as agent i. Since agents are identical in this sense, each of them has an equal right to the resource and hence must be treated equally and as a consequence agent i faces all possible orders of serving the agents with equal chance. For queueing problems, the notion of ICLB was analyzed by Chun and Yengin [9], Kayi and Ramaekers [22] and Mitra [26]. Following the notion of another type of participation constraint suggested by Gershkov and Schweinzer [18] for queueing problems, we define the ECLB for sequencing problems for which we consider the actual economy where each order is equally likely, that is, the participation constraint of any agent i is the expected cost in the actual economy when all possible orders of serving the agents is equally likely. Unlike ICLB, agents retain their differences in terms of processing time under ECLB.

For sequencing problems with no initial order, with our set of results we can identify a qualitative difference between ICLB and ECLB. Specifically, we show that for ECLB the agent specific restrictions in terms of processing times of the agents that are both necessary and sufficient to find mechanisms satisfying outcome efficiency, strategyproofness and participation constraints holds for all sequencing problems, with ICLB it holds only for a strict subset of sequencing problems. However, for the queueing problems, the notions of ICLB and ECLB coincide. Chun and Yengin [9] in the queueing context provided a necessary condition and a sufficient condition for obtaining mechanisms satisfying outcome efficiency, strategyproofness and ICLB (ECLB). Using our results we can generalize Chun and Yengin's [9] result by eliminating the gap between their necessary and sufficient conditions.

[3]See Bevia [1], Moulin [29], [30], Steinhaus [33] and Yengin [36].

2. THE FRAMEWORK

Consider a finite set of agents $N = \{1, 2, \ldots, n\}$ who want to process their jobs using a facility that can be used sequentially. The job processing time can be different for different agents. Specifically, for each agent $i \in N$, the job processing time is given by $s_i > 0$. Let $\theta_i S_i$ measure the cost of job completion for agent $i \in N$ where $S_i \in \mathbb{R}_{++}$ is the job completion time for this agent and $\theta_i \in \Theta := \mathbb{R}_{++}$ denotes his constant per-period waiting cost with $\mathbb{R}_{++}$ is the positive orthant of the real line $\mathbb{R}$. Due to the sequential nature of providing the service, the job completion time for agent i depends not only on his own processing time s_i, but also on the processing time of the agents who precede him in the order of service. By means of an order $\sigma = (\sigma_1, \ldots, \sigma_n)$ on N, one can describe the position of each agent in the order. Specifically, $\sigma_i = k$ indicates that agent i has the k-th position in the order. Let Σ be the set of $n!$ possible orders on N. For any order $\sigma \in \Sigma$, we define its *complement* order as σ^c, that is, σ^c is such that $\sigma_i^c = n + 1 - \sigma_i$ for all $i \in N$. We define $P_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j < \sigma_i\}$ to be the predecessor set of i in the order σ. Similarly, $F_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j > \sigma_i\}$ denotes the follower (or successor) set of i in the order σ. Note that for any $\sigma \in \Sigma$ and any $i \in N$, $P_i(\sigma) = F_i(\sigma^c)$ and $F_i(\sigma) = P_i(\sigma^c)$. Given a vector $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$ and an order $\sigma \in \Sigma$, the cost of job completion for agent $i \in N$ is $\theta_i S_i(\sigma)$, where the job completion time is $S_i(\sigma) = \sum_{j \in P_i(\sigma)} s_j + s_i$. In general, we use the following convention on the summation operator: for any set $Y = \{X_1, \ldots, X_K\}$ and any $M \subseteq Y$, $\sum_{j \in M} X_j = 0$ if $M = \emptyset$. The agents have quasi-linear utility of the form $u_i(\sigma, \tau_i; \theta_i) = -\theta_i S_i(\sigma) + \tau_i$ where σ is the order, $\tau_i \in \mathbb{R}$ is the transfer that he receives and the parameter of the model θ_i is the waiting cost. Given any processing time vector $s \in \mathbb{R}^n_{++}$, with slight abuse of notation, we denote a *sequencing problem* by Ω and we denote the set of all sequencing problems with the set of agents N by $\mathcal{S}(N)$. A sequencing problem $\Omega \in \mathcal{S}(N)$ is called a *queueing problem* if $s = (s_1, \ldots, s_n)$ is such that $s_1 = \ldots = s_n$. We denotes the set of all queueing problems with the set of agents N by $\mathcal{Q}(N)$. Clearly, $Q(N) \subset \mathcal{S}(N)$ for any given N (such that N is a finite set and $n \geq 2$).

A typical profile of waiting costs is denoted by $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$. For any $i \in N$, let θ_{-i}, denote the profile $(\theta_1 \ldots \theta_{i-1}, \theta_{i+1}, \ldots \theta_n) \in \Theta^{n-1}$ which is obtained from the profile θ by eliminating i's waiting cost. A mechanism is $\mu = (\sigma, \tau)$ that constitutes of a sequencing rule σ

and a transfer rule τ. A *sequencing rule* is a function $\sigma : \Theta^n \to \Sigma$ that specifies for each profile $\theta \in \Theta^n$ a unique order $\sigma(\theta) = (\sigma_1(\theta), \ldots, \sigma_n(\theta)) \in \Sigma$. Because the sequencing rule is a function (and not a correspondence) we will require a tie-breaking rule to reduce a correspondence to a function which, unless explicitly discussed, is assumed to be fixed. We use the following tie-breaking rule. We take the linear order $1 \succ 2 \succ \ldots \succ n$ on the set of agents N. For any sequencing rule σ and any profile $\theta \in \Theta^n$ with a tie situation between agents $i, j \in N$, we pick the order $\sigma(\theta)$ with $\sigma_i(\theta) < \sigma_j(\theta)$ if and only if $i \succ j$. A *transfer rule* is a function $\tau : \Theta^n \to \mathbb{R}^n$ that specifies for each profile $\theta \in \Theta^n$ a transfer vector $\tau(\theta) = (\tau_1(\theta), \ldots, \tau_n(\theta)) \in \mathbb{R}^n$. Specifically, given any mechanism $\mu = (\sigma, \tau)$, if (θ'_i, θ_{-i}) is the announced profile when the true waiting cost of i is θ_i, then utility of i is $u_i(\mu_i(\theta'_i, \theta_{-i}); \theta_i) = -\theta_i S_i(\sigma(\theta'_i, \theta_{-i})) + \tau_i(\theta'_i, \theta_{-i})$ where $\mu_i(\theta'_i, \theta_{-i}) := (\sigma(\theta'_i, \theta_{-i}), \tau_i(\theta'_i, \theta_{-i}))$. Given any $\Omega \in \mathcal{S}(N)$, any $\theta \in \Theta^n$ and any order $\sigma \in \Sigma$, define the aggregate cost as $C(\sigma; \theta)$, that is, $C(\sigma; \theta) := \sum_{j \in N} \theta_j S_j(\sigma)$.

Definition 1. A sequencing rule σ^* is *outcome efficient* if for any profile $\theta \in \Theta^n$, $\sigma^*(\theta) \in \text{argmin}_{\sigma \in \Sigma} C(\sigma; \theta)$.

The ratio of the waiting cost and processing time of any agent i, that is, θ_i / s_i is known as the urgency index. From Smith [32] it follows that σ^* is outcome efficient if and only if the following holds: **(OE)** For any $\theta \in \Theta^n$, the selected order $\sigma^*(\theta)$ satisfies the following: For any $i, j \in N$, $\theta_i / s_i > \theta_j / s_j \Leftrightarrow \sigma^*_i(\theta) < \sigma^*_j(\theta)$. We say that a mechanism $\mu = (\sigma, \tau)$ satisfies outcome efficiency if $\sigma = \sigma^*$.

Definition 2. For a sequencing rule σ, a mechanism $\mu = (\sigma, \tau)$ is *strategyproof* if the transfer rule $\tau : \Theta^n \to \mathbb{R}^n$ is such that for any $i \in N$, any $\theta_i, \theta'_i \in \Theta$ and any $\theta_{-i} \in \Theta^{n-1}$,

$$u_i(\mu_i(\theta); \theta_i) \geq u_i(\mu_i(\theta'_i, \theta_{-i}); \theta_i). \tag{1}$$

For a given sequencing rule σ, strategyproofness of a mechanism $\mu = (\sigma, \tau)$ requires that the transfer rule τ is such that truthful reporting for any agent weakly dominates false reporting no matter what others' report.

Definition 3. A mechanism μ satisfies *feasibility* if for any $\theta \in \Theta^n$, $\sum_{j \in N} \tau_i(\theta) \leq 0$.

Definition 4. A mechanism μ satisfies *budget balance* if for any $\theta \in \Theta^n$, $\sum_{j \in N} \tau_i(\theta) = 0$.

Clearly, for a sequencing rule σ, if the associated mechanism $\mu = (\sigma, \tau)$ is budget balanced, then it also feasible but the converse is not true.

2.1. **Participation constraints.** Given any sequencing problem $\Omega \in \mathcal{S}(N)$, let $-\theta_i O_i(s)$ be the participation constraint of agent i with type θ_i. Let $O(N;s) := (O_1(s), \dots, O_n(s)) \in \mathbb{R}^n$ denote the participation constraint vector. We represent a typical sequencing problem with participation constraints by $\Gamma = (\Omega, O(N;s))$ where $\Omega \in \mathcal{S}(N)$ and the associated $O(N;s) \in \mathbb{R}^n$ is the participation constraints vector.

Definition 5. For Γ, a mechanism $\mu = (\sigma, \tau)$ satisfies *participation constraints* if the transfer rule $\tau : \Theta^n \to \mathbb{R}^n$ is such that for any $i \in N$, any $\theta_i \in \Theta$ and any $\theta_{-i} \in \Theta^{n-1}$,

$$u_i(\mu_i(\theta_i, \theta_{-i}); \theta_i) \geq -\theta_i O_i(s). \tag{2}$$

3. PARTICIPATION CONSTRAINTS AND STRATEGYPROOFNESS

Given any sequencing problem with participation constraints Γ we first try to identify the set of all mechanisms that satisfy outcome efficiency, strategyproofness and participation constraints.

Definition 6. Any sequencing problem with participation constraints $\Gamma = (\Omega, O(N;s))$ satisfies the *interval* property if $O(N;s) = (O_1(s), \dots, O_n(s))$ is such that

$$O_i(s) \in [s_i, A(s)] \quad \forall\, i \in N \quad \text{where} \quad A(s) := \sum_{j \in N} s_j. \tag{3}$$

Let $\mathcal{G}(N)$ be the set of all Γ satisfying the interval property given by (3).

Theorem 1. The following statements are equivalent:

(SPC1) For a $\Gamma = (\Omega, O(N;s))$ we can find a mechanism that satisfies outcome efficiency, strategyproofness and participation constraints

(SPC2) Γ satisfies the interval property, that is, $\Gamma \in \mathcal{G}(N)$.

Proof: (SPC1) $\Rightarrow$ (SPC2) It is well-known that for an outcome efficient sequencing rule a mechanisms is strategyproof if and only if the associated transfer is a VCG transfer (see Holmström [21]). The standard way of specifying the VCG transfers for any sequencing problem

Ω is that for all $\theta \in \Theta^n$ and for all $i \in N$, $\tau_i(\theta) = -C(\sigma^*(\theta)), \theta) + S_i(\sigma^*(\theta))\theta_i + g_i(\theta_{-i})$, where for each $i \in N$ the function $g_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary.[4] If in addition we require participation constraints to be met, then it is necessary that for any profile $\theta \in \Theta^N$ and any agent $i \in N$, $U_i(\sigma^*(\theta), \tau_i(\theta); \theta_i) = -C(\sigma^*(\theta); \theta) + g_i(\theta_{-i}) \geq -\theta_i O_i(s)$ implying that $g_i(\theta_{-i}) \geq C(\sigma^*(\theta); \theta) - \theta_i O_i(s)$. Since the function $g_i(\theta_{-i})$ is independent of agent i's waiting cost θ_i, we have the following:

$$(4) \quad g_i(\theta_{-i}) \geq \bar{g}_i(\theta_{-i}) := \max_{x_i \in \Theta} \left[T_i(x_i; \theta_{-i})\right], \;\; T_i(x_i; \theta_{-i}) := \left[C(\sigma^*(x_i, \theta_{-i}); x_i, \theta_{-i}) - x_i O_i(s)\right].$$

Observe that $T_i(x_i; \theta_{-i}) = [S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)]x_i + \sum_{j \in N \setminus \{i\}} \theta_j S_j(\sigma^*(x_i, \theta_{-i}))$.

Consider any profile $\theta^* \in \Theta^n$ and any $i \in N$ such that $\theta^*_j / s_j = a > 0$ for all $j \in N \setminus \{i\}$. Consider any $x'_i, x''_i \in \Theta$ such that $x'_i/s_i \geq a \geq x''_i/s_i$ and $x'_i > x''_i$. We consider two cases-(a) $s_i > O_i(s)$ and (b) $A(s) < O_i(s)$. If $O_i(s) < s_i$, then we have

$$(5) \qquad T_i(x'_i; \theta^*_{-i}) - T_i(x''_i; \theta^*_{-i}) = (x'_i - x''_i)[s_i - O_i(s)] + \sum_{j \neq i} s_i s_j \left[\frac{\theta^*_j}{s_j} - \frac{x''_i}{s_i}\right] > 0.$$

Moreover, for any $x_i > s_i a$, $T_i(x_i; \theta^*_i) = x_i[s_i - O_i(s)] + \sum_{j \in N \setminus \{i\}} \theta^*_j S_j(\sigma^*(x_i, \theta^*_{-i}))$ is increasing in x_i. Therefore, the x^*_i that maximizes $T_i(x_i; \theta_{-i})$ is then $x^*_i = \infty$ implying that we do not have an interior maxima. Therefore, for the maximum to exist it is necessary that (a) $O_i(s) \geq s_i$. Moreover, if $O_i(s) > A(s) = \sum_{j \in N} s_j$, then we have

$$(6) \qquad T_i(x'_i; \theta^*_{-i}) - T_i(x''_i; \theta^*_{-i}) = (x'_i - x''_i)[A(s) - O_i(s)] + \sum_{j \neq i} s_i s_j \left[\frac{\theta^*_j}{s_j} - \frac{x'_i}{s_i}\right] < 0.$$

Similarly, for any $x_i < s_i a$, $T_i(x_i; \theta^*_i) = x_i[A(s) - O_i(s)] + \sum_{j \in N \setminus \{i\}} \theta^*_j S_j(\sigma^*(x_i, \theta^*_{-i}))$ is decreasing in x_i. Therefore, we cannot find an $x^*_i > 0$ that maximizes $T_i(x_i; \theta_{-i})$. Hence, we also require that (b) $O_i(s) \leq A(s)$. Combining (a) and (b) we get that $O_i(s) \in [s_i, A(s)]$. Since the selection of i was arbitrary, the result follows.

(SPC2) $\Rightarrow$ (SPC1) Consider any Γ that satisfies the interval property, that is, consider $\Gamma \in \mathcal{G}(N)$. We first argue how, using the interval property, the solution x^*_i to the maximization exercise of the function $T_i(x_i; \theta_{-i})$ can be made independent of the exact waiting cost of agent i.

[4] See Mitra [25] and Suijs [34].

Using this independence we then define a VCG mechanism for which participartion constraints can be satisfied.

For any profile $\theta \in \Theta^n$ and any $i \in N$, consider the type $x_i^* \in \Theta$ such that the function $T_i(x_i, \theta_{-i})$ (defined in condition (4)) takes the maximum value, that is, $T_i(x_i^*, \theta_{-i}) \geq T_i(x_i, \theta_{-i})$ for all $x_i \in \Theta^n$.

Step 1: For any $i \in N$ and any $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$, there exists $k \in N\setminus\{i\}$ such that $T_i(x_i^*; \theta_{-i}) = T_i(s_i(\theta_k/s_k); \theta_{-i})$.

Proof of Step 1: Consider any $i \in N$ and any $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$ and let $\tilde{R}(\theta_{-i}) = ((\tilde{R}_j(\theta_{-i}) = \theta_j/s_j))_{j\neq i})$ be the vector of agent specific waiting cost to processing time ratio of agents in $N \setminus \{i\}$ and $R(\theta_{-i}) = (R_1(\theta_{-i}) = \theta_{(1)}/s_{(1)}, \ldots, R_{n-1}(\theta_{-i}) = \theta_{(n-1)}/s_{(n-1)})$ be the permutation of $\tilde{R}(\theta_{-i})$ such that $R_1(\theta_{-i}) \geq \ldots \geq R_{n-1}(\theta_{-i})$.

We first show that there exists $x_i^* \in [s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ that maximizes $T_i(x_i, \theta_{-i})$. Observe that for any $x_i \in \Theta$, $T_i(x_i; \theta_{-i}) = [S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)]x_i + \sum_{j\in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i, \theta_{-i}))$. If $x_i > s_i R_1(\theta_{-i})$, then $S_i(\sigma^*(x_i, \theta_{-i})) = s_i$ and hence it follows that $T_i(x_i; \theta_{-i}) = [s_i - O_i(s)]x_i + \sum_{j\in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i, \theta_{-i}))$ which is non-increasing in x_i since by interval property $s_i \leq O_i(s)$ implying that the coefficient of x_i in $T_i(x_i; \theta_{-i})$ is non-positive. Hence, (a) if a maxima exists then we can always find a waiting cost $x_i^* \leq s_i R_1(\theta_{-i})$ that achieves it. Similarly, if $y_i < s_i R_{n-1}(\theta_{-i})$, then $S_i(\sigma^*(y_i, \theta_{-i})) = A(s)$ and hence it follows that $T_i(y_i; \theta_{-i}) = [A(s) - O_i(s)]y_i + \sum_{j\in N\setminus\{i\}} \theta_j S_i(\sigma^*(y_i, \theta_{-i}))$ which is non-decreasing in y_i since by interval property $A(s) \geq O_i(s)$ implying that the coefficient of x_i in $T_i(x_i; \theta_{-i})$ is non-negative. Hence, (b) if a maxima exists, then we can always find a waiting cost $x_i^* \geq s_i R_{n-1}(\theta_{-i})$ that achieves it.

The function $T_i(x_i; \theta_{-i})$ is continuous and concave in x_i on the interval $[s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ and the interval $[s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ is compact.[5] Hence, the function $T_i(x_i; \theta_{-i})$ has a maxima in $[s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$. Given $x_i^* \in [s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ and given continuity of $T_i(x_i; \theta_{-i})$, for two agents the proof is complete since $x_i^* = s_i R_1(\theta_j) = s_i(\theta_j/s_j)$ and it follows that $T_i(\theta_i(\theta_j), \theta_j) = [s_i - O_i(s)]s_i(\theta_j/s_j) + \theta_j(s_i + s_j)$. Therefore, consider the more than

[5]From the functional form of $T_i(x_i, \theta_{-i})$ and given outcome efficiency it is obvious that given any θ_{-i}, the function $T_i(x_i, \theta_{-i})$ is continuous in x_i on any open interval $(s_i R_{k+1}(\theta_{-i}), s_i R_k(\theta_{-i}))$ for all $k \in \{1, \ldots, n-2\}$ and by using appropriate limit argument one can also show continuity at any point $s_i R_k(\theta_{-i})$ for $k \in \{1, \ldots, n-1\}$. For concavity note that for any $\theta_{-i} \in \Theta_{-i}$, for every $x_i \in (s_i R_{k+1}(\theta_i), s_i R_k(\theta_i))$ for all $k \in \{0, \ldots, n\}$, where $R_{n+1} = 0$ and $R_0 = \infty$, $T_i(x_i, \theta_{-i}) = [S_i(\sigma^*(x_i, \theta_i)) - O_i(s)]\, x_i + \sum_{j\in N\setminus\{i\}} \theta_j s_j(\sigma^*(x_i, \theta_i))$ is a straight line. Moreover, $S_i(\sigma^*(x_i, \theta_i))$ is non-increasing in $x_i \in \mathbb{R}_{++}$. Hence the intercept $S_i(\sigma^*(x_i, \theta_i)) - O_i(s)$ is also non-increasing for $x_i \in \mathbb{R}_{++}$. As a result the piece-wise linear continuous function $T_i(x_i, \theta_{-i})$ is concave for $x_i \in \mathbb{R}_{++}$.

two agents case. If there exists $k \in N\backslash\{i\}$ such that $x_i^* = s_i(\theta_{(k)}/s_{(k)})$ (so that $T_i(x_i^*;\theta_{-i}) = T_i(s_i(\theta_k/s_k);\theta_{-i}) \geq T_i(x_i;\theta_{-i})$ holds for all $x_i \in \Theta$), then the proof is complete. If not then suppose there exists $k \in \{1,\ldots,n-2\}$ such that $x_i^* \in (s_i R_{k+1}(\theta_{-i}), s_i R_k(\theta_{-i}))$, that is,

$$T_i(x_i^*;\theta_{-i}) = \left[\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s)\right] x_i^* + \sum_{j \in N\backslash\{i\}} \theta_j S_j(\sigma^*(x_i^*,\theta_{-i})).$$

If $\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s) > 0$, then for any $x_i \in (x_i^*, s_i R_k(\theta_{-i})]$, $\sigma^*(x_i,\theta_{-i}) = \sigma^*(x_i^*,\theta_{-i})$ and $T_i(x_i;\theta_{-i}) > T_i(x_i^*;\theta_{-i})$ since $T_i(x_i;\theta_{-i}) - T_i(x_i^*;\theta_{-i}) = \left[\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s)\right](x_i - x_i^*) > 0$. Therefore we have a contradiction to our assumption that at x_i^* the function $T_i(x_i;\theta_{-i})$ is maximized. If $\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s) < 0$, then for any $x_i' \in [s_i R_k(\theta_{-i}), x_i^*)$, $\sigma^*(x_i',\theta_{-i}) = \sigma^*(x_i^*,\theta_{-i})$ and $T_i(x_i';\theta_{-i}) > T_i(x_i^*;\theta_{-i})$ since $T_i(x_i';\theta_{-i}) - T_i(x_i^*;\theta_{-i}) = \left[\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s)\right](x_i' - x_i^*) > 0$. Again we have a contradiction to our assumption that at x_i^* the function $T_i(x_i;\theta_{-i})$ is maximized. Therefore, the only possibility left is $\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s) = 0$. However, in that case $T_i(x_i^*;\theta_{-i}) = \sum_{j\in N\backslash\{i\}} \theta_j S_i(\sigma^*(x_i^*,\theta_{-i}))$ and for every $x_i \in [s_i R_{k+1}(\theta_{-i}), s_i R_k(\theta_{-i})]$ the function $T_i(x_i,\theta_{-i})$ attains its maximum value implying that $T_i(x_i^*;\theta_{-i}) = T_i(s_i R_{k+1}(\theta_{-i});\theta_{-i}) = T_i(s_i R_k(\theta_{-i});\theta_{-i})$ and Step 1 continues to be valid.

Given Step 1, let us define $x_i^* := \theta_i(\theta_{-i})$ so that $T_i(x_i^*;\theta_{-i}) = T_i(\theta_i(\theta_{-i});\theta_{-i})$ and there exists $k \in N\backslash\{i\}$ such that $\theta_i(\theta_{-i}) = s_i(\theta_k/s_k)$. To complete the proof, consider the VCG mechanism $\mu^* = (\sigma^*, \tau^*)$ with VCG transfers having the following property: For all $\theta \in \Theta^n$ and for all $i \in N$, $\tau_i^*(\theta) = -C(\sigma^*(\theta)),\theta) + S_i(\sigma^*(\theta))\theta_i + \bar{g}_i(\theta_{-i})$ with $\bar{g}_i(\theta_{-i}) := T_i(\theta_i(\theta_{-i});\theta_{-i})$. Then for any given $\theta \in \Theta^n$ and any agent $i \in N$, we have $u_i(\mu_i^*(\theta),\theta_i) + \theta_i O_i(s) = -[S_i(\sigma^*(\theta) - O_i(s)]\theta_i + \bar{g}_i(\theta_{-i}) = T_i(\theta_i(\theta_{-i}),\theta_{-i}) - T_i(\theta_i,\theta_{-i}) \geq 0$. The last inequality follows from the fact that $T_i(\theta_i,\theta_{-i}) \leq T_i(\theta_i(\theta_{-i}),\theta_{-i})$ for all $\theta_i \in \Theta$. Hence, $u_i(\mu_i^*(\theta),\theta_i) \geq -\theta_i O_i(s)$ implying that that the VCG mechanism μ^* satisfies the participation constraint for any agent i. Thus, using the interval property we have identified a VCG mechanism that satisfies participation constraints. □

Given any $\Gamma \in \mathcal{G}(N)$ what is the set of all mechanisms that satisfy outcome efficiency, strategyproofness and participation constraints? The next result answers this question. For any profile $\theta \in \Theta^n$, and any $i \in N$, define $X_i(\theta_{-i}) = \{(s_i\theta_j/s_j)_{j\in N\backslash\{i\}}\}$ as the set with each element as the product of urgency index an agent other than i and the processing time of agent i.

Proposition 1. For any $\Gamma \in \mathcal{G}(N)$, an outcome efficient mechanism $\mu^p = (\sigma^*, \tau^p)$ satisfies strategyproofness and participation constraints if and only if τ^p satisfies the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$,

$$\tau_i^p(\theta) = [S_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i})) - O_i(s)]\theta_i(\theta_{-i}) + \sum_{j \in N \setminus \{i\}} \theta_j \delta_{ji}(\theta) + h_i(\theta_{-i}), \tag{7}$$

where

$$\theta_i(\theta_{-i}) \in \arg \max_{x_i \in X_i(\theta_{-i})} T_i(x_i; \theta_{-i}) := [S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)]x_i + \sum_{j \in N \setminus \{i\}} \theta_j S_j(\sigma^*(x_i, \theta_{-i})), \tag{8}$$

$$\sum_{j \in N \setminus \{i\}} \theta_j \delta_{ji}(\theta) = \begin{cases} - \sum\limits_{j \in P_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}) \setminus P_i(\sigma^*(\theta))} \theta_j s_i & \text{if } P_i(\sigma^*(\theta)) \subset P_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}), \\ 0 & \text{if } P_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}) = P_i(\sigma^*(\theta)), \\ \sum\limits_{j \in P_i(\sigma^*(\theta)) \setminus P_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i})} \theta_j s_i & \text{if } P_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}) \subset P_i(\sigma^*(\theta)). \end{cases} \tag{9}$$

and $h_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}_+$.

Proof: For outcome efficiency and strategyproof it is necessary that the mechanism $\mu = (\sigma^*, \tau)$ must be VCG with transfers satisfying the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$, $\tau_i(\theta) = -C(\sigma^*(\theta); \theta) + g_i(\theta_{-i})$ where $g_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary. For participation constraint, in addition, it is necessary that (I) $g_i(\theta_{-i}) \geq \bar{g}_i(\theta_{-i}) = T_i(\theta_i(\theta_{-i}); \theta_{-i}) \in \max_{x_i \in \Theta} T_i(x_i; \theta_{-i})$ and $T_i(x_i; \theta_{-i}) = [S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)]x_i + \sum_{j \in N \setminus \{i\}} \theta_j S_j(\sigma^*(x_i, \theta_{-i}))$ (see condition (4) in the proof of Theorem 1). Hence, using (I) we can replace $g_i(\theta_{-i}) = h_i(\theta_{-i}) + T_i(\theta_i(\theta_{-i}); \theta_{-i})$ where $h_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ and $h_i(\theta_{-i}) \geq 0$. By substituting $g_i(\theta_{-i}) = h_i(\theta_{-i}) + T_i(\theta_i(\theta_{-i}); \theta_{-i})$ in the transfer $\tau_i(\theta)$ and then simplifying it we get

$$\tau_i(\theta) = [S_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i})) - O_i(s)]\theta_i(\theta_{-i}) + \sum_{j \in N \setminus \{i\}} \theta_j \delta_{ji}(\theta) + h_i(\theta_{-i}), \tag{10}$$

where $\delta_{ji}(\theta) := \left(\sum_{k \in P_j(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}))} s_k - \sum_{k \in P_j(\sigma^*(\theta))} s_k \right)$. Observe the following:

(a) If $P_j(\sigma^*(\theta)) = P_j(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}))$, then $\delta_{ji}(\theta) = 0$.

(b) If $P_j(\sigma^*(\theta)) \subset P_j(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}))$, then $P_j(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i})) \setminus P_j(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i})) = \{i\}$ and $\delta_{ji}(\theta) = s_i$.

(c) If $P_j(\sigma^*(\theta_i(\theta_{-i}),\theta_{-i})) \subset P_j(\sigma^*(\theta))$, then $P_j(\sigma^*(\theta)) \setminus P_j(\sigma^*(\theta_i(\theta_{-i}),\theta_{-i})) = \{i\}$ and $\delta_{ji}(\theta) = -s_i$.

By substituting the values of $\delta_{ji}(\theta)$ for possibilities (a), (b) and (c) in the sum $\sum_{j\in N\setminus\{i\}} \theta_j \delta_{ji}(\theta)$ we get (9).[6] From (I) condition (10) and the expansion of the sum $\sum_{j\in N\setminus\{i\}} \theta_j \delta_{ji}(\theta)$ given by (9) we get $\tau = \tau^p$.

To prove the converse, observe that since any μ^p is a particular type of VCG transfers, μ^p is sufficient to ensure outcome efficency and strategyproofness. To complete the proof we need to check the sufficiency of participation constraints with μ^p. Consider any μ^p. For any $\theta \in \Theta^n$ and any $i \in N$,
$u_i(\mu_i^p(\theta),\theta_i) + \theta_i O_i(s)$
$= -\theta_i[S_i(\sigma^*(\theta)) - O_i(s)] + [S_i(\sigma^*(\theta_i(\theta_{-i}),\theta_{-i})) - O_i(s)]\theta_i(\theta_{-i}) + \sum_{j\in N\setminus\{i\}} \theta_j \delta_{ji}(\theta) + h_i(\theta_{-i})$
$= T_i(\theta_i(\theta_{-i}),\theta_{-i}) - T_i(\theta) + h_i(\theta_{-i}) \geq 0$. Therefore, $u_i(\mu_i^p(\theta),\theta_i) + \theta_i O_i(s) \geq 0$ implying $u_i(\mu_i^p(\theta),\theta_i) \geq -\theta_i O_i(s)$. Hence, μ^p satisfies participation constraints. □

4. Application 1: Sequencing problems with initial order

For a sequencing problem $\Omega \in \mathcal{S}(N)$ with initial order, there is a preexisting order in which the agents have arrived to use the facility. Suppose that initial order of arrival is $\sigma^0 \in \Sigma$. In this case, the participation constraints vector is $O^{\sigma_0}(N,s) = (O_1^{\sigma_0}(s),\dots,O_n^{\sigma_0}(s)) \in \mathbb{R}^n_{++}$ where for each $i \in N$, $O_i^{\sigma_0}(s) = s_i + \sum_{j\in P_i(\sigma^0)} s_j$ and hence for any profile $\theta \in \Theta^n$, $\sum_{j\in N} \theta_j O_j^{\sigma_0}(s) = C(\sigma^0,\theta)$. Let $\mathcal{I}(N) = \{(\Omega, O^{\sigma_0}(N,s)) \mid \Omega \in \mathcal{S}(N), \sigma^0 \in \Sigma\}$ denote the set of all sequencing problems with initial order and let Γ^0 represent a typical sequencing problem with initial order in $\mathcal{I}(N)$. Every $\Gamma^0 \in \mathcal{I}(N)$ satisfies the interval property. For any $\Gamma^0 \in \mathcal{I}(N)$ with initial order σ^0, the participation constraints vector $O^{\sigma_0}(N,s) = (O_1^{\sigma_0}(s),\dots,O_n^{\sigma_0}(s)) \in \mathbb{R}^n_{++}$ is such that for each $i \in N$, $O_i^{\sigma_0}(s) = s_i + \sum_{j\in P_i(\sigma^0)} s_j$. Observe that for each $i \in N$, $O_i^{\sigma_0}(s) = s_i + \sum_{j\in P_i(\sigma^0)} s_j \in [s_i, A(s)]$ implying that the interval property condition (3) holds for every agent $i \in N$.

Proposition 2. For any $\Gamma^0 = (\Omega, O^{\sigma_0}(N,s)) \in \mathcal{I}(N)$, an outcome efficient mechanism $\mu^I = (\sigma^*, \tau^I)$ satisfies strategyproofness and participation constraints if and only if τ^I satisfies the

[6] The $sign(x)$ function is defined as follows: $sign(x) = 1$ if $x > 0$, $sign(x) = 0$ if $x = 0$ and $sign(x) = -1$ if $x < 0$.

following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$,

$$\tau_i^I(\theta) = \left[\sum_{j \in P_i(\sigma^*(\theta_i(\theta_{-i}),\theta_{-i}))} s_j - \sum_{j \in P_i(\sigma^0)} s_j\right] \theta_i(\theta_{-i}) + \sum_{j \in N\setminus\{i\}} \theta_j \delta_{ji}(\theta) + h_i^I(\theta_{-i}), \tag{11}$$

where $\delta_{ji}(\theta) = sign\left(|P_j(\sigma^*(\theta_i(\theta_{-i}),\theta_{-i}))| - |P_j(\sigma^*(\theta))|\right) s_i$ for all $j \in N \setminus \{i\}$, $h_i^I : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}_+$, $\theta_i(\theta_{-i}) \in \arg\max_{x_i \in X_i(\theta_{-i})} T_i^I(x_i;\theta_{-i})$ and

$$T_i^I(x_i;\theta_{-i}) = \left[\sum_{j \in P_i(\sigma^*(x_i,\theta_{-i}))} s_j - \sum_{j \in P_i(\sigma^0)} s_j\right] x_i + \sum_{j \in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i,\theta_{-i})). \tag{12}$$

Proof: Given Proposition 1, we only need derive the explicit form of $[S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)]x_i + \sum_{j\in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i,\theta_{-i}))$ in this context. Note that given $O_i^{\sigma^0}(s) = S_i(\sigma^0) = s_i + \sum_{j\in P_i(\sigma^0)} s_j$, $S_i(\sigma^*(x_i,\theta_{-i})) - O_i^{\sigma^0}(s) = \{s_i + \sum_{j\in P_i(\sigma^*(x_i,\theta_{-i})} s_j\} - \{s_i + \sum_{j\in P_i(\sigma^0)} s_j\} = \sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))} s_j - \sum_{j\in P_i(\sigma^0)} s_j$. By substituting the expression in the coefficient of x_i in $T_i(x_i;\theta_{-i})$ we get $T_i^I(x_i;\theta_{-i})$ given by (12) and we also get the coefficient of the first term $\theta_i(\theta_{-i})$ in (11) by substituting $x_i = \theta_i(\theta_{-i})$. $\square$

Remark 1. Consider any $\Gamma^0 = (\Omega, O^{\sigma_0}(N,s)) \in \mathcal{I}(N)$. We provide certain observations about the mechanisms $\mu^I = (\sigma^*, \tau^I)$ given by conditions (11) and (12).

(I1) Let $k \in N$ be that agent having last queueing position under that initial order σ^0, that is, $S_i(\sigma^0) = A(s) = \sum_{j\in N} s_j$. Then for any $\theta \in \Theta^n$, $\theta_k(\theta_{-k}) = s_k.\{\min\{\theta_j/s_j\}_{j\in N\setminus\{k\}}\}$ and $P_k(\sigma^*(\theta_k(\theta_{-k}),\theta_{-k})) = P_i(\sigma^0) = N \setminus \{k\}$. The reason being that for any $x_k \in X_k(\theta_{-k}) \setminus \{\theta_k(\theta_{-k})\}$, $P_k(\sigma^*(x_k,\theta_{-k})) \subset N \setminus \{n\}$ and hence $T_k^I(x_k;\theta_{-k}))$ is decreasing in x_k since $[\sum_{j\in P_k(\sigma^*(x_k,\theta_{-k}))} s_j - \sum_{j\in P_k(\sigma^0)} s_j] = \sum_{j\in P_k(\sigma^*(x_k,\theta_{-k}))\subset N\setminus\{k\}} s_j - \sum_{j\in N\setminus\{k\}} s_j$ is negative. Given $\theta_k(\theta_{-k}) = s_k.\{\min\{\theta_j/s_j\}_{j\in N\setminus\{k\}}\}$, it is quite easy to verify that (b1) $[\sum_{j\in P_k(\sigma^*(\theta_k(\theta_{-k}),\theta_{-k}))} s_j - \sum_{j\in P_k(\sigma^0)} s_j]\theta_k(\theta_{-k}) = 0$ and (b2) $\delta_{jk}(\theta) = -1$ for all $j \in F_k(\sigma^*(\theta))$ and $\delta_{jk}(\theta) = 0$ for all other $j \in P_k(\sigma^*(\theta))$. Therefore, using (b1) and (b2) in (11) we get

$$\tau_k^I(\theta) = -s_k \sum_{j\in F_k(\sigma^*(\theta))} \theta_j + h_k^I(\theta_{-k}).$$

(I2) Let $i \in N$ be that agent having first queueing position under that initial order σ^0, that is, $S_i(\sigma^0) = s_i$. Then for any profile $\theta \in \Theta^n$, $\theta_i(\theta_{-i}) = s_i.\{\max\{\theta_j/s_j\}_{j\in N\setminus\{i\}}\}$ and $P_i(\sigma^*(\theta_i(\theta_{-i}),\theta_{-i})) = P_i(\sigma^0) = \emptyset$. The reason being that for any $x_i \in X_i(\theta_{-i}) \setminus$

$\{\theta_i(\theta_{-i})\}$, $P_i(\sigma^*(x_i,\theta_{-i})) \neq \emptyset$ and hence $T_i^I(x_i;\theta_{-i}))$ is increasing in x_i since the coefficient of x_i, that is $[\sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))} s_j - \sum_{j\in P_i(\sigma^0)} s_j] = \sum_{j\in P_i(\sigma^*(x_1,\theta_{-1}))} s_j$ is positive. Given $\theta_i(\theta_{-i}) = s_i.\{\max\{\theta_j/s_j\}_{j\in N\setminus\{i\}}\}$, (a1) $[\sum_{j\in P_i(\sigma^*(\theta_i(\theta_{-i}),\theta_{-i}))} s_j - \sum_{j\in P_i(\sigma^0)} s_j]\theta_i(\theta_{-i}) = 0$ and (a2) $\delta_{ji}(\theta) = 1$ for all $j \in P_i(\sigma^*(\theta))$ and $\delta_{ji}(\theta) = 0$ for all other $j \in F_i(\sigma^*(\theta))$. Therefore, using (a1) and (a2) in (11) we get

$$\tau_i^I(\theta) = s_i \sum_{j\in P_i(\sigma^*(\theta))} \theta_j + h_i^I(\theta_{-i}).$$

(I3) For any profile $\theta \in \Theta^n$ such that $\sigma^*(\theta) = \sigma^0$, it is easy to verify that $P_i(\sigma^*(\theta_i(\theta_{-i}),\theta_{-i})) = P_i(\sigma^*(\theta)) = P_i(\sigma^0)$ for all $i \in N$. Hence, using (11) it follows that $\tau_i^I(\theta) = h_i^I(\theta_{-i}) \geq 0$ for all $i \in N$.

(I4) Consider the mechanism $\bar{\mu}^I = (\sigma^*, \bar{\tau}^I)$ satisfying (11) and (12) and for this mechanism $\bar{h}_i^I(\theta_{-i}) = 0$ for all $\theta_{-i} \in \Theta^{n-1}$ and for all $i \in N$. Observe that for any other mechanism $\mu^I = (\sigma^*, \tau^I)$ satisfying conditions (11) and (12) we have the following property: For any $\theta \in \Theta^n$, $\sum_{j\in N} \tau_j^I(\theta) - \sum_{j\in N} \bar{\tau}_j^I(\theta) = \sum_{j\in N} h_j^I(\theta_{-j}) \geq 0$. Hence, if a mechanism $\mu^I = (\sigma^*, \tau^I)$ is feasible, then the mechanism $\bar{\mu}^I = (\sigma^*, \bar{\tau}^I)$ is also feasible. Hence to analyze feasibility, we only use the mechanism $\bar{\mu}^I$ in the next result.

Proposition 3. For any $\Gamma^0 = (\Omega, O^{\sigma_0}(N,s)) \in \mathcal{I}(N)$, there is no mechanism that satisfies outcome efficiency, strategyproofness, participation constraints and feasibility.

Proof: Consider any $\Gamma^0 = (\Omega, O^{\sigma_0}(N,s)) \in \mathcal{I}(N)$ and, without loss of generality, assume σ^0 such that $\sigma_i^0 = i$ for all $i \in N$. Consider any $\theta \in \Theta^n$ such that $\theta_n/s_n > \theta_1/s_1 > \ldots > \theta_{n-1}/s_{n-1}$ so that $P_1(\sigma^*(\theta)) = \{n\}$, $P_j(\sigma^*(\theta)) = \{1,\ldots,j-1\} \cup \{n\}$ for all $j \in N \setminus \{1,n\}$ and $P_n(\sigma^*(\theta)) = \emptyset$. Fix, the mechanism $\bar{\mu}^I = (\sigma^*, \bar{\tau}^I)$ (defined in condition (I4) of Remark 1). It is easy to verify the following:

(i) Given $P_n(\sigma^0) = N \setminus \{n\}$, from condition (I1) of Remark 1 we get $\theta_n(\theta_{-n}) = s_n\theta_{n-1}/s_{n-1}$ and $P_n(\sigma^*(\theta_n(\theta_{-n}),\theta_{-n})) = P_n(\sigma^0) = N \setminus \{n\}$. Moreover, we also have $P_j(\sigma^*(\theta)) \setminus P_j(\sigma^*(\theta_n(\theta_{-n}),\theta_{-n})) = \{n\}$ for all $j \in N \setminus \{n\}$. Hence, the transfer of n is $\bar{\tau}_n^I(\theta) = \sum_{j\in N\setminus\{n\}} \theta_j\delta_{jn}(\theta) = -\sum_{j\in N\setminus\{n\}} \theta_j s_n$. Therefore, the transfer of agent n does not involve the waiting cost θ_n.

(ii) Given $P_1(\sigma^0) = \emptyset$, from condition (I2) of Remark 1 we have $\theta_1(\theta_{-1}) = s_1\theta_n/s_n$ and $P_1(\sigma^*(\theta_1(\theta_{-1}), \theta_{-1})) = P_1(\sigma^0) = \emptyset$. Further, $P_n(\sigma^*(\theta_1(\theta_{-1}), \theta_{-1})) \setminus P_n(\sigma^*(\theta)) = \{1\}$ and $P_j(\sigma^*(\theta_1(\theta_{-1}), \theta_{-1})) = P_j(\sigma^*(\theta))$ for all $j \in N \setminus \{1, n\}$. Thus, $\bar{\tau}^I_1(\theta) = \theta_n\delta_{n1}(\theta) = \theta_n s_1$.

(iii) Finally, consider any $k \in N \setminus \{1, n\}$. Observe that if $x_k = s_k\theta_n/s_n$, then $T^I_k(x_k; \theta_{-k}))$ is decreasing in x_k since the coefficient of x_k, that is $[\sum_{j\in P_k(\sigma^*(x_k,\theta_{-k}))} s_j - \sum_{j\in P_k(\sigma^0)} s_j] = -\sum_{j=1}^{k-1} s_j$ is negative. Therefore, $\theta_k(\theta_{-k}) \neq s_k\theta_n/s_n$. Moreover, one can easily verify that $\delta_{nk}(\theta) = 0$ since $P_n(\sigma^*(\theta_k(\theta_{-k}), \theta_{-k})) = P_n(\sigma^*(\theta)) = \emptyset$. Therefore, the transfer of any agent k does not involve the waiting cost θ_n and hence is of the form $\bar{\tau}^I_k(\theta) = \theta_k(\theta_{-k})[\sum_{j\in P_k(\sigma^*(x_k,\theta_{-k}))} s_j - \sum_{j\in P_k(\sigma^0)} s_k] + \sum_{j\in N\setminus\{k,n\}} \theta_j\delta_{jk}(\theta)$.

From (i), (ii) and (iii) it follows that $\sum_{j\in N} \bar{\tau}^I_j(\theta) = \theta_n s_1 + \sum_{j\in N\setminus\{1\}} \bar{\tau}^I_j(\theta)$. From (i) and (iii) above it also follows that the sum $\sum_{j\in N\setminus\{1\}} \bar{\tau}^I_j(\theta)$ does not involve the waiting cost θ_n and hence by defining $\mathcal{T}(\sigma^*(\theta); \theta_{-n}) := \sum_{j\in N\setminus\{1\}} \bar{\tau}^I_j(\theta)$ we get

$$\sum_{j\in N} \bar{\tau}^I_j(\theta) = \theta_n s_1 + \mathcal{T}(\sigma^*(\theta); \theta_{-n}). \tag{13}$$

If $\sum_{j\in N} \bar{\tau}^I_j(\theta) > 0$, then we have a violation of feasibility and the proof is complete. Therefore, assume $\sum_{j\in N} \bar{\tau}^I_j(\theta) = \theta_n s_1 + \mathcal{T}(\sigma^*(\theta); \theta_{-n}) \leq 0$. Given that $\mathcal{T}(\sigma^*(\theta); \theta_{-n})$ is independent of θ_n, if we increase the waiting cost of agent n to any $y_n (> \theta_n)$ by keeping θ_{-n} fixed, then the outcome efficient order remains unchanged (that is, $\sigma^*(y_n, \theta_{-n}) = \sigma^*(\theta)$ for all $y_n > \theta_n$) and the transfers of all but agent 1 continues to remain unchanged due to this independent, that is, $\mathcal{T}(\sigma^*(y_n, \theta_{-n}); \theta_{-n}) = \mathcal{T}(\sigma^*(\theta); \theta_{-n})$ for all $y_n > \theta_n$. Hence, we have

$$\sum_{j\in N} \bar{\tau}^I_j(y_n, \theta_{-n}) = y_n s_1 + \mathcal{T}(\sigma^*(\theta); \theta_{-n}) \ \ \forall\, y_n > \theta_n. \tag{14}$$

Since the first term in the right hand side of condition (14) is increasing in y_n and the second term is a constant, for y_n sufficiently large (say y^*_n) we have $\sum_{j\in N} \bar{\tau}^I_j(y^*_n, \theta_{-n}) = y^*_n s_1 + \mathcal{T}(\sigma^*(\theta); \theta_{-n}) > 0$ leading to a violation of feasibility. $\square$

5. Application 2: Sequencing problems without initial order

For a sequencing problem $\Omega \in \mathcal{S}(N)$ without initial order, there is no preexisting order in which the agents have arrived to use the facility. Therefore, what is a reasonable notion of participation constraints in this scenario?

5.1. **Identical costs lower bound.** Identical costs lower bound (ICLB) requires that each agent receives at least the utility he could expect under the egalitarian solution if all agents were like him in a reference economy. In the sequencing context, given an agent i, if all other agents are identical to agent i, then agent i has an equal chance of facing each order from Σ. This means that all agents are identical to agent i in the sense that every agent has the same waiting cost and processing time in the reference economy. Thus, ICLB requires that for any agent $i \in N$ and any profile $\theta \in \Theta^n$, $u_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq -\theta_i((n+1)s_i/2)$ where $\theta_i((n+1)s_i/2)$ represents the expected cost of agent i with waiting cost θ_i and processing time s_i when all agents have the same processing time s_i and agent i gets each of the queue positions 1 to n with probability $1/n$.

For a sequencing problem $\Omega \in \mathcal{S}(N)$ with participation constraints given the identical costs lower bounds, the participation constraints vector is $O^s(N,s) = (O_1^{s_1}(s), \ldots, O_n^{s_n}(s)) \in \mathbb{R}^n_{++}$ where for each $i \in N$, $O_i^{s_i}(s) = (n+1)s_i/2$. Let $\overline{\mathcal{C}}(N) = \{(\Omega, O^s(N,s)) \mid \Omega \in \mathcal{S}(N)\}$ denote the set of all sequencing problems with ICLB and let Γ^s represent a typical sequencing problem with ICLB in $\overline{\mathcal{C}}(N)$. It is easy to verify that a sequencing problem $\Gamma^s \in \overline{\mathcal{C}}(N)$ satisfies the interval property if and only if (C) for any $i \in N$, $A(s) \geq (n+1)s_i/2$. Let $\mathcal{C}(N) (\subset \overline{\mathcal{C}}(N))$ denote the set of all problems $\Gamma^s \in \overline{\mathcal{C}}(N)$ that also satisfies condition (C).

Proposition 4. For any $\Gamma^s \in \mathcal{C}(N)$, an outcome efficient mechanism $\mu^C = (\sigma^*, \tau^C)$ satisfies strategyproofness and participation constraints if and only if τ^C satisfies the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$,

$$\tau_i^C(\theta) = \left[\sum_{j \in P_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}))} s_j - \frac{(n-1)s_i}{2}\right] \theta_i(\theta_{-i}) + \sum_{j \in N \setminus \{i\}} \theta_j \delta_{ji}(\theta) + h_i^C(\theta_{-i}), \tag{15}$$

where $\theta_i(\theta_{-i}) \in \arg\max_{x_i \in \Theta} T_i^C(x_i;\theta_{-i})$, the sum $\sum_{j \in N\setminus\{i\}} \theta_j \delta_{ji}(\theta)$ is given by condition (9), $h_i^C : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$, $h_i^C(\theta_{-i}) \geq 0$ and

$$T_i^C(x_i;\theta_{-i}) = \left[\sum_{j \in P_i(\sigma^*(x_i,\theta_{-i}))} s_j - \frac{(n-1)s_i}{2}\right] x_i + \sum_{j \in N\setminus\{i\}} \theta_j S_i(\sigma^*(x_i,\theta_{-i})). \tag{16}$$

Proof: Given Proposition 1, we only need derive the explicit form of the term $[S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)]$ in this context. Given $O_i^s(s) = (n+1)s_j/2$, it easily follows that $S_i(\sigma^*(x_i,\theta_{-i})) - O_i^{s_i}(s) = \{s_i + \sum_{j \in P_i(\sigma^*(x_i,\theta_{-i})} s_j\} - \{(n+1)s_i/2\} = \sum_{j \in P_i(\sigma^*(x_i,\theta_{-i}))} s_j - (n+1)s_i/2$. By substituting the expression in the coefficient of x_i in $T_i(x_i;\theta_{-i})$ we get $T_i^C(x_i;\theta_{-i})$ given by (16) and we also get the coefficient of the first term $\theta_i(\theta_{-i})$ in (15) by substituting $x_i = \theta_i(\theta_{-i})$. $\square$

Clearly, we can find $\Gamma^s \in \mathcal{C}(N)$ such that condition (C) is violated implying that outcome efficiency, strategyproofness and participation constraints are incompatible. As an example, consider any $\Omega \in \mathcal{S}(N)$ and its associated $\Gamma^s = (\Omega, O^s(N,s))$ such that there exists $i \in N$ such that $s_i/2 > s_j$ for all $j \in N \setminus \{i\}$. In this case, $O_i(s) = (n+1)s_i/2 = s_i + \sum_{j \neq i}(s_i/2) > s_i + \sum_{j\neq i} s_j = A(s)$. Therefore, we have a violation of interval property that requires that $O_i(s) \leq A(s)$. Given this incompatibility, in the next subsection, we consider another notion of participation constraints for such sequencing problems without initial order that can completely eliminate these instances of incompatibility.

5.2. **Expected cost lower bounds.** In the queueing context, Gershkov and Schweinzer [18] defined another type of participation constraints which we call the "expected cost lower bound" (ECLB). ECLB considers the actual economy where each order is equally likely and, unlike the notion of ICLB, agents retain their difference in terms of processing time. Then, to meet the condition associated with ECLB, we need the following property: For any agent $i \in N$ and any profile $\theta \in \Theta^n$, $u_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq -\theta_i \left(\sum_{\sigma \in \Sigma} \frac{S_i(\sigma)}{n!}\right)$. Define $\bar{S}_i := s_i + \sum_{j \in N\setminus\{i\}}(s_j/2)$ for each $i \in N$. It is quite easy to verify that for each agent $i \in N$, $\sum_{\sigma \in \Sigma} \frac{S_i(\sigma)}{n!} = \bar{S}_i$.[7] Therefore, an

[7]The equality $\sum_{\sigma \in \Sigma} \frac{S_i(\sigma)}{n!} = \bar{S}_i$ states that the average completion time of each agent i equals $\bar{S}_i$. The sum in $\bar{S}_i$ has two components-own processing time s_i and half of the total processing time of all other agents $j \neq i$. In any possible ordering $\sigma \in \Sigma$, an agent will always incur his own processing time and hence s_i enters $\bar{S}_i$ with probability one. Moreover, observe that any other agent $j \neq i$ precedes agent i in any ordering σ if and only if he does not precede agent i in the complement ordering σ^c. Therefore, when we consider all possible orderings to calculate agent i's average completion time, s_j for $j \neq i$ will occur in exactly half of the cases as a part of the completion time of agent i.

equivalent representation of the ECLB requirement is that for any agent $i \in N$ and any profile $\theta \in \Theta^n$, $u_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq -\theta_i \bar{S}_i$.

For a sequencing problem $\Omega \in \mathcal{S}(N)$ with participation constraints given the ECLB conditions, the participation constraints vector is $O^{\bar{S}}(N, s) = (O_1^{\bar{S}_1}(s), \ldots, O_n^{\bar{S}_n}(s)) \in \mathbb{R}^n_{++}$ where for each $i \in N$, $O_i^{\bar{S}_i}(s) = \bar{S}_i$. Let $\mathcal{E}(N) = \{(\Omega, O^{\bar{S}}(N, s)) \mid \Omega \in \mathcal{S}(N)\}$ denote the set of all sequencing problems with ECLB and let $\Gamma^{\bar{S}}$ represent a typical sequencing problem with ECLB in $\mathcal{E}(N)$.

Remark 2. For any queueing problem $\Omega \in \mathcal{Q}(N)$ with $s_1 = \ldots = s_n = a > 0$, $\bar{S}_i = (n+1)a/2$ for all $i \in N$ implying that the notions of ICLB and ECLB are equivalent. Clearly, the bounds associated with ICLB and ECLB are different for any sequencing problem which is not a queueing problem, that is for any $\Omega \in \mathcal{S}(N) \setminus \mathcal{Q}(N)$. Specifically, ECLB allows for agents to be treated as identical only in terms of waiting costs but, unlike ICLB, not in terms of processing time. This makes sense since in our context processing time of the agents are common knowledge. Unlike ICLB, all sequencing problem with ECLB as its participation constraints satisfy the interval property, that is for every $\Gamma^{\bar{S}} \in \mathcal{E}(N)$ the interval property condition (3). Observe that for any $\Gamma^{\bar{S}} \in \mathcal{E}(N)$ and any $i \in N$, $O_i(s) = \bar{S}_i = s_i + \sum_{j \in N \setminus \{i\}} (s_j/2) \in (s_i, A(s))$ implying that the interval property given by condition (3) holds.

Proposition 5. For any $\Gamma^{\bar{S}} \in \mathcal{E}(N)$, an outcome efficient mechanism $\mu^E = (\sigma^*, \tau^E)$ satisfies strategyproofness and participation constraints if and only if τ^E satisfies the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$,

$$\tau_i^E(\theta) = \left[\sum_{k \in P_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}))} \frac{s_k}{2} - \sum_{k \in F_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}))} \frac{s_k}{2} \right] \theta_i(\theta_{-i}) + \sum_{j \in N \setminus \{i\}} \theta_j \delta_{ji}(\theta) + h_i^E(\theta_{-i}), \tag{17}$$

where $\theta_i(\theta_{-i}) \in \arg\max_{x_i \in \Theta} T_i^E(x_i; \theta_{-i})$, the sum $\sum_{j \in N \setminus \{i\}} \theta_j \delta_{ji}(\theta)$ is given by condition (9), $h_i^E : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$, $h_i^E(\theta_{-i}) \geq 0$ and

$$T_i^E(x_i; \theta_{-i}) = \left[\sum_{k \in P_i(\sigma^*(x_i, \theta_{-i}))} \frac{s_k}{2} - \sum_{k \in F_i(\sigma^*(x_i, \theta_{-i}))} \frac{s_k}{2} \right] x_i + \sum_{j \in N \setminus \{i\}} \theta_j S_i(\sigma^*(x_i, \theta_{-i})). \tag{18}$$

Proof: Given Proposition 1, we only need derive the explicit form of $[S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)]$ in this context. Note that given $O_i^{\bar{S}_i}(s) = \bar{S}_i = s_i + \sum_{j \in N \setminus \{i\}} (s_j/2)$, $S_i(\sigma^*(x_i, \theta_{-i})) - O_i^{\bar{S}_i}(s) = \{s_i +$

$\sum_{j\in P_i(\sigma^*(x_i,\theta_{-i})} s_j\} - \{s_i + \sum_{j\in N\setminus\{i\}}(s_j/2)) = \sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))}(s_j/2) - \sum_{j\in F_i(\sigma^*(x_i,\theta_{-i}))}(s_j/2)$. By substituting the expression in the coefficient of x_i in $T_i(x_i;\theta_{-i})$ we get $T_i^E(x_i;\theta_{-i})$ given by (18) and we also get the coefficient of the first term $\theta_i(\theta_{-i})$ in (17) by substituting $x_i = \theta_i(\theta_{-i})$. □

5.3. **Implications in terms of queueing problems.** Throughout this subsection we assume without loss of generality that $s_1 = \ldots = s_n = 1$. First consider the function $T_i^C(x_i;\theta_{-i})$ given in condition (16) of Proposition 4. For the queueing problem, for any $\theta_{-i} \in \Theta^n$, for any $i \in N$ and for any $x_i \in \Theta$ we have

$$T_i^C(x_i;\theta_{-i}) = [|P_i(\sigma^*(x_i,\theta_{-i}))| - (n-1)/2]x_i + \sum_{j\in N\setminus\{i\}} \sigma_j^*(x_i,\theta_{-i})\theta_j. \tag{19}$$

Similarly, consider the function $T_i^E(x_i;\theta_{-i})$ given in condition (18) of Proposition 5. For the queueing problem, for any $\theta_{-i} \in \Theta^n$, for any $i \in N$ and for any $x_i \in \Theta$ we have (A) $T_i^E(x_i;\theta_{-i}) = [|P_i(\sigma^*(x_i,\theta_{-i}))|/2 - |F_i(\sigma^*(x_i,\theta_{-i}))|/2]x_i + \sum_{j\in N\setminus\{i\}} \sigma_j^*(x_i,\theta_{-i})\theta_j$. Using $|F_i(\sigma^*(x_i,\theta_{-i}))|/2 = (n-1-|P_i(\sigma^*(x_i,\theta_{-i}))|)/2$ and then simplifying (A) we get the following: For the queueing problem, for any $\theta_{-i} \in \Theta^n$, for any $i \in N$ and for any $x_i \in \Theta$ we have

$$T_i^E(x_i;\theta_{-i}) = [|P_i(\sigma^*(x_i,\theta_{-i}))| - (n-1)/2]x_i + \sum_{j\in N\setminus\{i\}} \sigma_j^*(x_i,\theta_{-i})\theta_j = T_i^C(x_i;\theta_{-i}). \tag{20}$$

Hence, from (19) and (20) it follows that for the queueing problem, for any $\theta_{-i} \in \Theta^n$, for any $i \in N$ and for any $x_i \in \Theta$, $T_i^E(x_i;\theta_{-i}) = T_i^C(x_i;\theta_{-i})$.

Definition 7. For σ^* and for any positive integer $K \le |N|$, a mechanism $\mu^k = (\sigma^*, \tau^{(K)})$ is a *K-pivotal mechanism* if for any $\theta \in \Theta^n$ and any $i \in N$,

$$\tau_i^{(K)}(\theta) = \begin{cases} -\sum\limits_{j:\sigma_i^*(\theta)<\sigma_j^*(\theta)\le K} \theta_j & \text{if } \sigma_i^*(\theta) < K, \\ 0 & \text{if } \sigma_i^*(\theta) = K, \\ \sum\limits_{j:K\le\sigma_j^*(\theta)<\sigma_i^*(\theta)} \theta_j & \text{if } \sigma_i^*(\theta) > K. \end{cases} \tag{21}$$

See Mitra and Mutuswami [27] who introduce and characterize the K-pivotal mechanisms for the queueing problems. Chun and Yengin [9] also provide another characterization of these mechanism. We define a new set of mechanisms which are obtained by appropriately mixing different K-pivotal mechanisms.

Definition 8. For any queueing problem, a mechanism $\bar{\mu}^a = (\sigma^*, \bar{\tau}^a)$ is a *centered K-pivotal mechanism with non-negative intercepts* if for all $\theta \in \Theta^n$ and all $i \in N$,

$$\bar{\tau}_i^a(\theta) = H_i(\theta_{-i}) + \begin{cases} \tau_i^{\left(\frac{n+1}{2}\right)}(\theta) & \text{if } n \text{ is odd,} \\ \frac{1}{2}\tau_i^{\left(\frac{n}{2}\right)}(\theta) + \frac{1}{2}\tau_i^{\left(\frac{n}{2}+1\right)}(\theta) & \text{if } n \text{ is even,} \end{cases} \tag{22}$$

where for each $i \in N$, the function $H_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is such that $H_i(\theta_{-i}) \geq 0$ for all $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$.

Corollary 1. For any queueing problem $\Omega \in \mathcal{Q}(N)$, a mechanisms satisfies outcome efficiency strategyproofness and ICLB (ECLB) if and only if it is a centered K-pivotal mechanism with non-negative intercepts.

Proof: For any profile $\theta \in \Theta^n$ and any $i \in N$, consider the type $\theta_i(\theta_{-i}) \in \Theta$ such that the function $T_i(x_i, \theta_{-i})$ (defined in condition (20)) takes the maximum value, that is, $T_i(\theta_i(\theta_{-i}), \theta_{-i}) \geq T_i(x_i, \theta_{-i})$ for all $x_i \in \Theta^n$. Let $\bar{r}(\theta_{-i}) = ((\bar{r}_j(\theta_{-i}) = \theta_j))_{j \neq i})$ be the vector of agent specific waiting cost agents in $N \setminus \{i\}$ and $r(\theta_{-i}) = (r_1(\theta_{-i}) = \theta_{(1)}, \ldots, r_{n-1}(\theta_{-i}) = \theta_{(n-1)})$ be the permutation of $\bar{r}(\theta_{-i})$ such that $r_1(\theta_{-i}) \geq \ldots \geq r_{n-1}(\theta_{-i})$. One can verify that if n is odd, then $\theta_i(\theta_{-i}) = r_{\frac{(n-1)}{2}}(\theta_{-i})$ so that $|P_i(\sigma^*(x_i, \theta_{-i}))| = (n-1)/2$. Then using condition (16) (condition 18) of Proposition 4 (Proposition 5) we get

$$\tau_i(\theta) = \sum_{j \in N \setminus \{i\}} \theta_j \delta_{ji}(\theta) + H_i(\theta_{-i}), \tag{23}$$

where $H_i(\theta_{-i}) \geq 0$ and $\sum_{j \in N \setminus \{i\}} \theta_j \delta_{ji}(\theta)$ is given by condition (9) of Proposition 1. A simplification of $\sum_{j \in N \setminus \{i\}} \theta_j \delta_{ji}(\theta)$ using condition (9) gives the following:

$$\sum_{j \in N \setminus \{i\}} \theta_j \delta_{ji}(\theta) = \tau_i^{\left(\frac{n+1}{2}\right)}(\theta) = \begin{cases} -\sum\limits_{j: \sigma_i^*(\theta) < \sigma_j^*(\theta) \leq \frac{n+1}{2}} \theta_j & \text{if } \sigma_i^*(\theta) < \frac{n+1}{2}, \\ 0 & \text{if } \sigma_i^*(\theta) = \frac{n+1}{2}, \\ \sum\limits_{j: \frac{n+1}{2} \leq \sigma_j^*(\theta) < \sigma_i^*(\theta)} \theta_j & \text{if } \sigma_i^*(\theta) > \frac{n+1}{2}. \end{cases} \tag{24}$$

From (25) and (24), the result follows for the case when n is odd.

If n is even, then $\theta_i(\theta_{-i}) \in \{r_{\frac{n}{2}-1}(\theta_{-i}), r_{\frac{n}{2}(\theta_{-i})}\}$. Using condition (9) one can verify that whether we select $\theta_i(\theta_{-i}) = r_{\frac{n}{2}-1}(\theta_{-i})$ or we select $\theta_i(\theta_{-i}) = r_{\frac{n}{2}}(\theta_{-i})$ does not really matter since the transfer that we get under both cases are identical and is given by

$$\tau_i(\theta) = \frac{1}{2}\tau_i^{\left(\frac{n}{2}\right)}(\theta) + \frac{1}{2}\tau_i^{\left(\frac{n}{2}+1\right)}(\theta) + H_i(\theta_{-i}), \tag{25}$$

where $H_i(\theta_{-i}) \geq 0$. Hence, we get the result for n even. □

Remark 3. Chun and Yengin [9] in the queueing context provided a necessary and a sufficient condition for obtaining mechanisms satisfying outcome efficiency, strategyproofness and ICLB. In Chun and Yengin [9], there is no gap between the necessary and sufficient conditions when the number of agents n is odd and (like in Corollary 1) it is the standard $((n+1)/2)$-pivotal mechanism up to a non-negative agent specific constant functions $H_i(.)$. However, there is a gap when $n \geq 4$ is even. Specifically, Chun and Yengin's [9] necessary condition on transfers gives a $(n/2+1)$-pivotal mechanism up to a non-negative agent specific constant function $H_i(.)$ and their sufficient condition on transfers gives a $n/2$-pivotal mechanism up to a non-negative agent specific constant function $H_i(.)$. Corollary 1 shows that the simple average between the necessary and sufficient mechanisms in Chun and Yengin [9] uniquely characterizes the set of all strategyproof, ICLB mechanisms with outcome efficienct sequencing rule.

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