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Structural Manipulation in a Multi-period Sender-receiver Game*

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Abstract

This paper shows how an informed politician can manipulate uninformed, yet "rational," citizens in a multi-period sender-receiver game. By examining the payoff structure, the citizens find out that the politician has an incentive to be honest. Hence, they never doubt him and make their decision according to his messages. However, the information structure restricts the citizens' ability to detect his misleading them, and so they cannot punish him properly. Therefore, under this game structure, the politician can mislead them without having significant disadvantages, and hence will manipulate their decision in his favor. This phenomenon is referred to as *structural manipulation*.

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1 Introduction

History, such as the Mukden incident in 1931 (Duus, 1989), Korean Red-bait plot in 1996 (Cho, 2000), and the Russian apartment bombings in 1999 (Litvinenko and Felshtinsky, 2007), shows that, by sending misleading information, an informed politician can manipulate uninformed citizens in his or her favor. But, when the uninformed citizens are rational, is this still possible? If so, how can an informed politician manipulate uninformed, yet rational, citizens?

To see this possibility, we simplify those historical events to the following multi-period sender-receiver game in which a politician (sender) sends a message and multiple citizens (receivers) make a joint decision. The game lasts for many periods, and each period is characterized by one of two states: a Peace state (P-state) in which an opponent country supports peace unless they are invaded and a War state (W-state) in which the opponent country supports war. To make this game economically interesting, we assume that the P-state would most likely occur, but the W-state can also occur with small, but positive, probability. In each period, the politician observes the actual state and sends the citizens one of two messages: a Peace message (P-message) by which he denotes that the P-state has occurred and a War message (W-message) by which he denotes that the W-state has occurred. After observing his message, but without information on the actual state, the citizens make a joint decision on whether to support peace (P-decision) or to support war (W-decision). Regarding their preferences, the politician prefers the W-decision to the P-decision regardless of the states, while the citizens prefer the P-decision in the P-state and prefer the W-decision in the W-state.

In this simplified game, the citizens are rational. Hence, they clearly understand that the politician can mislead them whenever he can benefit from it. As such, they do not naively or credulously believe his messages. However, the citizens also know that the game lasts for a lot of periods. Hence, the politician can have huge payoffs in the future, but he will lose them if he sends untruthful messages and the citizens find it out. Accordingly, they rationally conclude that he will not put his huge future payoffs at risk by sending untruthful messages. Therefore, under the payoff structure of this game, they will "never rationally doubt" the veracity of his messages, and as a consequence, they will make their joint decision according to his messages; namely, when the politician sends the P-message, they think that the P-state has occurred and thus will make the P-decision, which is their favorite in the P-state, and likewise, when he sends the W-message, they will make the W-decision, which is their favorite in the W-state.

The payoff structure of this game is common knowledge. Hence, the politician knows that the citizens know that he has the huge future payoffs and, if he sends untruthful messages and they find it out, he will lose them. Accordingly, he can figure out that they will never rationally doubt the veracity of his messages and thus will make their decision according to his messages. Consequently, he realizes that, by sending the W-message, he can induce them to make the W-decision, which is his favorite regardless of the states. He can think further that, once the citizens make the W-decision, it will trigger defensive action by the opponent country, and eventually both countries will be involved in war. Hence, in this case, the citizen cannot find out the actual state because, regardless of the actual state, they will wind up with the same outcome, the war against their opponent country. So, the politician also realizes that, under the information structure of this game, he can even secure his future payoffs without a significant disadvantage from sending misleading information, which in turn leads him to actually commit it. Therefore, in this game, the politician sends misleading information, and successfully manipulates the "rational" citizens in his favor, even against their own interests.

This paper extends the simplified game above to cover a general situation of information transmission, and proposes a theory that explains how an informed player (politician) can manipulate a joint decision by a group of uninformed, yet rational, players (citizens). The theory shows that the game structure, related to both the payoff and the information, plays a key role in his manipulating the uninformed players, as briefly shown in the game above. Under the payoff structure, the uninformed players choose their actions according to the informed player's messages. Then, the informed player strategically sends his messages and induces them to choose his own favorite. (For example, in the game above, the politician sends only the W-message regardless of the states and induces the citizens to make the W-decision, which is his favorite.) Once the uninformed players choose the informed player's favorite, they cannot figure out the actual state due to the information structure. (For example, in the game above, once the citizens make the W-decision, they cannot figure out the actual state, because it will lead to the same outcome, the war with their opponent country, regardless of the actual state.) Hence, without a significant disadvantage from sending untruthful messages, the informed player can successfully manipulate the rational uninformed players in his favor. Therefore, the game structure itself enables the informed player to manipulate the rational uninformed players in his favor. We refer to this phenomenon as *Structural Manipulation*. The goal of the paper is to promote understanding of this structural manipulation.

- Structure for the structural manipulation

The structural manipulation can arise in various economic environments because the structure, which gives rise to this phenomenon, can capture a wide range of economic situations. The payoff structure similar to that in this paper, where the informed player is possibly a long-run player and hence has concern for his future payoffs, naturally appears in a multi-period game without further assumptions, such as in Sobel (1985), Benabou and Laroque (1992), Morris (2001), and Ely and Välimäki (2003). On the other hand, the information structure similar to that in this paper, where the uninformed players cannot always figure out the actual state and so the informed player can mislead them without having a significant disadvantage, requires some additional assumptions.

For example, in the simplified game above, the citizens can figure out the actual state if and only if they make the P-decision. This is because, if they make the P-decision, they will wind up with different outcomes in different states; namely, a peace outcome in the P-state and a war outcome in the W-state. Hence, by examining the outcome, the citizens can figure out the actual state. But, if they make the W-decision, they will wind up with the same outcome, war outcome, regardless of the actual state, and hence cannot figure out the actual state. Therefore, once they make the W-decision, which is the politician's favorite, they cannot find out whether or not the politician sends misleading information, and thus cannot punish him properly.

This kind of information structure, however, has been widely used in the economics literature, such as Kreps and Wilson (1982), Baliga and Sjöström (2001), and Jung (2009).[1] In particular, it could be a natural setting in the following three situations.

First, when the informed player diagnoses his customer's problem and then makes a recommendation, this information structure could be a natural setting. For example, as in Ely and Välimäki (2003), consider a motorist (uninformed player) who has a car problem. An auto mechanic (informed player) first diagnoses the problem and makes a recommendation. For simplicity, suppose that there are only two states: an E-state in which the car needs a new $Engine$ and a T-state in which it needs a mere $Tune-up$. The mechanic gets higher repair fees by replacing the engine than by tuning up. Accordingly, he prefers the engine replacement to the tune-up. In this situation, if the motorist (she) chooses a tune-up, her car will be fixed in the T-state, but it will not in the E-state. Hence, by examining her car, she can figure out the actual state. On the other hand, if she chooses an engine replacement, her car will be fixed regardless of the original problem. Hence, she cannot figure out the actual state. Therefore, if the mechanic successfully induces her to choose the engine replacement, then, even when he does it by sending misleading information, he will not have a significant disadvantage from it, because the motorist cannot actually find it out. A similar logic applies in many contexts, such as medical examination, management consultant, and legal counsel.[2]

[1] See also Fudenberg and Levine (1989), Aoyagi (1996), Celetani et al. (1996), Morris (2001), Ely and Välimäki (2003), Baliga and Sjöström (2004), Cripps et al. (2005), and Dellarocas (2006).

[2] A medical doctor could diagnose a patient who needs either serious medical treatment (large fees) or light medical treatment (small fees). A management consultant could have a firm manager whose firm needs either a major reorganization or a minor reform. Finally, a lawyer could meet a client who has either a complex case that needs legal assistance of the lawyer or a simple case that does not. Note that Ely and Välimäki (2003)

Second, when the informed player has multiple types and one of the types has a dominant strategy, this information structure naturally appears. For example, as in Kreps and Wilson (1982), suppose there are two players: a Monopolist (informed player) and an Entrant (uninformed player). The monopolist can be either *strong* or *weak*. The strong monopolist has a dominant strategy: *Fight in every period*. In this situation, if the monopolist does not play *Fight* in some period, the entrant can be certain that the monopolist is weak. On the other hand, if he plays only *Fight* in every period, the entrant cannot figure out his type. Therefore, in this game, by persistently playing *Fight*, the weak monopolist can successfully mislead the entrant. Likewise, a good deal of the literature on reputation effects can exemplify this situation.

Finally, when the informed player first reviews projects or products and then makes a report on them, it could also be a natural setting. For example, suppose that there are two players, agent (informed player) and principal (uninformed player), and two projects, project A and project B.[3] The agent first assesses the qualities of the projects and then reports on the result to the principal. Later, the principal will implement one of the two projects. We assume that the principal (she) can figure out the true quality of a project if and only if she actually implements the project. In this game, consider the case in which the agent prefers, say, project A to project B, and hence makes an incorrect and worst assessment of project B, while making a correct (but possibly not very good) assessment of project A. If the principal believes the report and implements project A, she cannot figure out the true quality of project B, and hence will not find out that the agent has sent misleading information. Therefore, the agent can still mislead the principal without a significant disadvantage. A similar logic can apply in business consultancy, the mining industry, and consumer reports.

- Related literature

This paper is closely related to Morris (2001) and Ely and Välimäki (2003). Morris (2001) also builds on a multi-period sender-receiver game, and shows that a sender's concern for future payoffs could adversely affect the payoff to a receiver, as in our paper. Ely and Välimäki (2003) present a concrete model that clearly shows the adverse effect of this concern. Their papers and this paper, however, show different effects of the sender's concern for future payoffs; namely, in their papers, it makes receivers less dependent on the sender, while, in this paper, it makes the uninformed players more dependent on the informed player. Specifically, they assume that a sender (he) has two possible types: good and bad. Each type of sender will have higher payoffs in the future if the receiver believes that he is good. Accordingly, the sender has an incentive not to make an impression that he is bad, and thus avoids

introduce these examples.

[3] This example is adapted from Baliga and Sjöström (2001).

sending messages that are typically sent by a bad type sender. Hence, even a good type sender conveys less information, which makes the sender's messages less informative and thus less useful to the receiver. Therefore, the sender's concern for his future payoffs makes the receiver less dependent on him, which leads the receiver to choose an inefficient action, and consequently, both the sender and the receiver become worse off. In this paper, on the other hand, the informed player's concern for his future payoffs makes the uninformed players never rationally doubt the veracity of his messages, and hence they make their joint decision according to his messages. Therefore, his concern makes the uninformed players more dependent on him, which leads them to choose only his favorite, and consequently, he becomes better off while the uninformed players become worse off.

The economics literature has explored the possibility of manipulation in various contexts. Benabou and Laroque (1992) adapt Sobel's (1985) model for asset markets and show that, if a sender has imperfect information about the actual state, the sender could repeatedly manipulate asset prices by releasing strategically distorted information. Baliga and Sjöström (2012) build on the conflict game of Baliga and Sjöström (2004) and show that an informed agent with extreme agendas could manipulate conflicts between decision makers by sending a public message that inflames tensions between them.[4] The logic behind their manipulation results, however, differs from that in this paper. In Benabou and Laroque (1992), with positive probability (but not with probability one), the sender truthfully reports whatever he or she observes, and, in Baliga and Sjöström (2012), the informed agent partially reveals some information about the actual state in the process of the manipulation. Hence, their messages are informative to the receivers. As a result, the receivers use the messages to update their priors and make optimal decisions accordingly, which consequently shows how the sender can have influence over the receivers. In this paper, on the other hand, the informed player's messages would not be informative to the uninformed players. (For example, in the simplified game, the politician sends only the W-message regardless of the actual state, and hence his messages do not contain any information about the actual state.) Nevertheless, due to the structure of the game, the informed player can have influence over the uninformed players.

The rest of the paper is organized as follows. Section 2 defines the formal model. Section 3 presents the results, including the weakest sufficient condition and the strongest necessary condition for the structural manipulation. Section 4 concludes with a discussion about how to solve the structural manipulation problem. All proofs are relegated to the Appendix.

[4] See also Mirman et al. (1994), Bueno de Mesquita (2010), and Edmond (2013).

2 Model

Consider the situation discussed in the introduction. We model this situation as a multi-period sender-receiver game.

The game lasts for T periods, and T can be either finite or infinite. Each period $t \leq T$ is characterized by a *state* (of the world), denoted by θ. The states are random variables on $[0, 1]$ that are independently and identically distributed according to a probability measure μ. For simplicity, we assume that the measure μ is absolutely continuous and the set $[0, 1]$ is its smallest closed support.

In each period, one *politician* (informed player) and two *citizens* (uninformed players), denoted by citizen 1 and citizen 2, play the game.[5] Note that this setting of the two citizens can be replaced with that of finitely many citizens, while preserving the same results as in this paper. We employ the current setting for simplicity.

The politician is possibly a long-run player in that he could continue to play the game. However, if he has sent an untruthful message and the citizens do not play *zeros*, the incumbent politician will be replaced with a new one at the end of the period, and the new one will play the game in the next period.[6] In contrast, the citizens are short-run players in that they play the game only for one period. Hence, each period has new citizens who have never played the game before. Contrary to our current model, if the citizens were long-run players and patient enough, they could collude with each other effectively, as shown by folk theorems, and thus could make the politician send only truthful messages. Celentani and Pesendorfer (1996), however, explain that if there are a large number of long-run players, then, since no individual player can effectively influence the history, they will be strategically myopic. Accordingly, the result in this paper remains valid even when we replace the current simple setting with that of a continuum of long-run citizens.

The game proceeds as follows. 1) In period $t \leq T$, an incumbent politician observes a

[5] Jung (2009) considers a simple model in which there is only one receiver, and shows that a sender can still manipulate the receiver's action in his favor.

[6] In most countries, this condition is stipulated by the Constitution or the law. First, intentionally spreading false information is against the law. In the U.S., for example, U.S.C. Title 18 Section 1038 says "whoever engages in any conduct with intent to convey false or misleading information ... be fined ... or imprisoned ..." As a consequence, this violation can result in discharge from his or her position. For example, U.S. Constitution Article 2 Section 4 says "the president, vice president and all civil officers of the United States, shall be removed from office on impeachment for, and conviction of ... high crimes and misdemeanors" and U.S. Constitution Article 1 Section 5 Clause 2 says "each House may ... punish its members for disorderly behavior, and, with the concurrence of two-thirds, expel a Member." In addition, some licensed senders, such as medical doctors, lawyers, and stockbrokers, are required to follow high standards of ethics. Reporting misleading information is strictly prohibited and thus can result in suspension or revocation of their licenses. In the U.S., for example, each state has a medical board, and each board has the authority to suspend or revoke the medical license of any physician who violates medical ethics. Likewise, an attorney disciplinary board and the SEC (Securities and Exchange Commission) supervise the lawyers and the stockbrokers, respectively.

state θ. 2) Then, he sends two citizens a (possibly untruthful) *message* $m \in [0,1]$, and his message is viewed as a literal statement about the state θ.[7] 3) After observing the politician's message m, but without information on the actual state θ, the two citizens simultaneously and independently take their *actions* $a = (a_1, a_2) \in \mathbb{R}_+ \times \mathbb{R}_+$, where we denote by a_1 citizen 1's action, by a_2 citizen 2's action, and by $\mathbb{R}_+$ the set of non-negative reals. The citizens may acquire the information about prior moves and payoffs realized in the previous periods, but this information is irrelevant to their optimal actions in the current period. 4) After all actions are taken, their current period payoffs are realized. 5) If period t is the last period, that is, $t = T$, the game ends at the end of period t. If period t is not the last period, that is, $t < T$, at the end of period t, the two incumbent citizens are always replaced by new ones. However, the incumbent politician is replaced by a new one if and only if he has sent an untruthful message, which means $m \neq \theta$, and the citizens do not play *zeros*. In the next period $t+1$, the same process repeats.

The payoff structure of this game is defined as follows. The *politician's discount factor* is denoted by $\delta \in (0,1]$, and *his single period payoff function* is a continuously differentiable function $u : \mathbb{R}_+^2 \longrightarrow \mathbb{R}$, where $u(a)$ denotes his single period payoff when the citizens take the actions a ($\in \mathbb{R}_+^2$). The politician maximizes the discounted sum of his single period expected payoffs. Assumption 1 below defines his preferences over the outcomes.

Assumption 1. Given any $a_j > 0$, $\partial u(a_i, a_j)/\partial a_i < 0$ for each $i \in \{1, 2\}$.[8]

Assumption 1 states that the politician strictly prefers the citizens' lower actions in $\mathbb{R}_+^2$ to their higher actions, that is, if $a > a'$, then $u(a) < u(a')$. Hence, the politician has a consistent incentive to influence the citizens to play lower actions regardless of the states, and naturally there is no partitional equilibrium as in Crawford and Sobel (1982). Assumption 1 is qualitatively the same as that for the bad adviser in Morris (2001) and that for the bad mechanic in Ely and Välimäki (2003). Based on this assumption, we can measure the politician's influence over the citizens. That is, the politician can be viewed as more influential if he can induce the citizens to play lower actions.

Each citizen i maximizes her expected payoff. Both her action a_i, according to a function $\nu : [0,1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}$, and the other citizen's action a_j, according to another function $w : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, contribute to citizen i's payoff. To model the information structure under which the citizens can figure out the actual state if and only if they play actions other than the politician's favorite, we define their payoff functions as simple products of the two contribution functions and assume that $w(a_j) = 0$ if and only if $a_j = 0$.

[7] That is, we assume that each message has its own intrinsic meaning, and the issues about how the meanings are endogenously formed under the standard cheap talk assumption are beyond the scope of the paper.

[8] Throughout the paper, we assume that i, $j \in \{1, 2\}$ and $i \neq j$.

More precisely, *citizen i's payoff function* $w \cdot \nu$ is defined as

$$w(a_j) \cdot \nu(\theta, a_i) = a_j \cdot \left[\theta^2 - \{\theta - (\theta_0 + a_i)\}^2\right],$$

where θ_0 is a positive real number. In this model, the citizens cannot observe the actual state. They can, however, observe their payoffs. With this payoff function, conditional on playing positive actions, they have different payoffs in different states. Hence, as long as they play positive actions, they can figure out the actual state by examining their payoffs. In contrast, if they play *zeros*, they will get *zeros* regardless of the actual state. Hence, they cannot figure out the actual state. Note that, according to Assumption 1, the politician's favorite actions are *zeros*. Therefore, this payoff function ensures that the citizens cannot figure out the actual state if and only if they both play the politician's favorite actions.

This payoff function $w \cdot \nu$ greatly simplifies the analysis by making the other citizen j's contribution w irrelevant to citizen i's action a_i. Hence, given a state, to find citizen i's optimal action, which maximizes her payoff $w \cdot \nu$, we need to take into account only her own contribution ν. For expositional convenience, if citizen i is indifferent among her actions, we assume that she will choose the lowest action among them. Then, for every state θ, since $\partial^2\nu/\partial a_i^2 < 0$ and $\partial^2\nu/\partial\theta\partial a_i > 0$, the optimal action $a_i(\theta)$, from the standpoint of a fully informed citizen, is uniquely defined.[9]

In addition, we restrict attention to only pure strategy equilibria. In equilibrium, the politician may play a mixed strategy, whereas the citizens always play pure strategies since $\partial^2\nu/\partial a_i^2 < 0$ and $\partial^2\nu/\partial\theta\partial a_i > 0$. Given any mixed strategy equilibrium, however, we can construct a pure strategy equilibrium that contains the same (pure) strategies of the citizens as in the mixed strategy equilibrium.[10] Therefore, this restriction does not constrain the conditions for the structural manipulation, and only simplifies the analysis.

Finally, to make the model economically interesting, we add two more assumptions. First, we exclude from consideration, by assuming away, the trivial case in which the politician can

[9] We can expend the functions w and v while preserving the results. Suppose that the functions w and v are continuously differentiable and satisfy the four conditions: 1. $w(a_j) = 0$ if and only if $a_j = 0$; 2. given any action $a_i > 0$, $\partial\nu(\cdot, a_i)/\partial\theta > 0$; 3. $\partial\nu(0,0)/\partial a_i < 0$ and $\partial\nu(1, a_i')/\partial a_i = 0$ for some action $a_i' > 0$; and 4. $\partial\nu(\theta, a_i')/\partial a_i < \partial\nu(\theta, a_i)/\partial a_i$ and $\partial\nu(\theta', a_i)/\partial a_i > \partial\nu(\theta, a_i)/\partial a_i$ for any actions $a_i' > a_i$ and any states $\theta' > \theta$. Then, all the results in this paper remains unchanged.

[10] To see this, suppose that a strategy profile Φ constitutes an equilibrium and it is mixed in that under Φ, the politician sends two distinct messages m and m' with positive probabilities in some states. Note that he can send the different messages m and m' only when those messages induce the same actions of the citizens, which also means that these messages induce the same conditional expected values of the state. Now, suppose that Φ' is another strategy profile and it is obtained from Φ only by switching from his sending m' to his sending m. Then, when the politician plays the game according to the strategy in Φ', in response to the message m, the optimal actions of the citizens are still the same, since it will induce the same conditional expected value of the state. Accordingly, Φ' constitutes an equilibrium. We can iterate this procedure, and can construct a pure strategy equilibrium that contains the same pure strategies of the citizens as Φ.

achieve his favorite outcome because of the analogy between his preferences over the outcomes and the citizens'. That is, we assume that the citizens' preferences are significantly different from the politician's. Formally, let $a([0,1])$ denote the citizens' actions that maximize their expected payoffs throughout the whole states $[0,1]$. Then, this assumption means that $a([0,1]) > (0,0)$. Second, we normalize the politician's payoff after he is dismissed from his (incumbent politician) position to be *zero*, and we restrict attention to the cases in which the politician can always secure himself positive expected payoffs as long as he plays the game as an incumbent politician. Formally, this assumption means that $u(a([0,1])) > 0$.

3 Results

This section presents the results. We employ perfect Bayesian equilibrium, formulated by Fudenberg and Tirole (1991), as our equilibrium concept. Note that whenever clear from the context, the reference to the fact that a result holds almost surely will be omitted. Proposition 1 first establishes the existence of a perfect Bayesian equilibrium in the model.

Proposition 1 *There exists an equilibrium.*

Next, Subsection 3.1 presents a sufficient condition for the structural manipulation (Theorem 1), and shows that this condition is the weakest sufficient condition (Proposition 2). Subsection 3.2 presents a necessary condition (Theorem 2), and shows that this condition is the strongest necessary condition (Proposition 3).

3.1 Sufficient condition

The sufficient condition is closely related to the citizens' optimal action functions, so we first define these functions. These functions specify the citizens' optimal actions as a function of a set of states. More formally, let $\beta[0,1]$ denote the Borel subsets of $[0,1]$. *Citizen i's optimal action function* $a_i : \beta[0,1] \longrightarrow \mathbb{R}_+$ is defined as follows. Given a set of states $\Theta \in \beta[0,1]$, if $\mu(\Theta) > 0$, $a_i(\Theta)$ is defined as $\inf\{a_i \in \mathbb{R}_+ : \partial \int_\Theta \nu(a_i,\theta)d\mu/\partial a_i = 2\int_\Theta \theta d\mu - 2(\theta_0 + a_i)\mu(\Theta) \leq 0\}$. If Θ is a singleton (and thus $\mu(\Theta) = 0$), that is, there exists a state θ' such that $\Theta = \{\theta'\}$, then $a_i(\Theta)$ is defined as $\inf\{a_i \in \mathbb{R}_+ : \partial\nu(a_i,\theta')/\partial a_i = 2\{\theta' - (\theta_0 + a_i)\} \leq 0\}$. Finally, if Θ is not a singleton and $\mu(\Theta) = 0$, $a_i(\Theta)$ is defined as a non-negative real between $a_i(\{0\})$ and $a_i(\{1\})$. Then, the optimal action function $a_i(\cdot)$ is well-defined and, for each $\Theta \in \beta[0,1]$, $a_i(\Theta)$ denotes citizen i's action such that, throughout the states Θ, $a_i(\Theta)$ maximizes her expected payoff. In addition, we define *the citizens' optimal action functions* $a : \beta[0,1] \longrightarrow \mathbb{R}^2_+$ as $a(\Theta) = (a_1(\Theta), a_2(\Theta))$ for each $\Theta \in \beta[0,1]$. To ease notational burden, when a set Θ is a singleton, we drop the braces $\{\ \}$ from the optimal action functions. That is, when Θ

$= \{\theta'\}$ for some state θ', we denote $a_i(\Theta)$ not by $a_i(\{\theta'\})$, but by $a_i(\theta')$ and we denote $a(\Theta)$ not by $a(\{\theta'\})$, but by $a(\theta')$.

Now, we are ready to present our first main result, the sufficient condition for the structural manipulation, where, due to the structure of the game, the politician can manipulate the citizens' actions in his favor. Note that, according to Assumption 1, the politician's favorite outcome is *zeros*. Hence, the structural manipulation results in the unique equilibrium outcome *zeros*.

Theorem 1 *If the inequality $u(a(1)) + \delta u(a([0,1])) \geq u(a(0))$ holds, an equilibrium outcome in each period $t < T$ is unique, and it is zeros.*

Remark 1 *Theorem 1 means that, if the politician has a sufficiently high future payoff, then, in every period t except the last period T, the citizens will play only the politician's favorite outcome zeros in equilibrium. The prerequisite for the uniqueness result is specified by the inequality $u(a(1)) + \delta u(a([0,1])) \geq u(a(0))$. In this inequality, the left-hand side, $u(a(1)) + \delta u(a([0,1]))$, is the infimum of the politician's possible payoffs when he sends truthful messages and secures his future payoff. The right-hand side, $u(a(0))$, is the supremum of his possible payoffs when he sends untruthful messages and ruins his future payoff. Hence, this inequality $u(a(1)) + \delta u(a([0,1])) \geq u(a(0))$ implies that the politician can have a higher payoff by doing the former than that by doing the latter due to his future payoff, and consequently, it captures the situation in which he has a sufficiently high future payoff. Note that the result in this Theorem is derived under the assumption that the citizens are rational. In addition, note that, contrary to our current model, if there were no politician, the citizens could play positive actions in equilibrium and could improve their payoffs. Therefore, Theorem 1 evidences that, when the citizens are rational, if the politician has a sufficiently high future payoff, he can manipulate their actions in his favor, even against their own interests.*

The intuition behind Theorem 1 is as follows. The citizens' strategies can be categorized into two groups: one group that includes strategies under which the citizens do not play his favorite outcome *zeros* in response to any of his messages (that is, they play only positive actions regardless of his messages), and the other that includes strategies under which they play *zeros* in response to some of his messages. For simplicity, the strategies in the former are referred to as *doubt strategies*, and those in the latter *non-doubt strategies*. In response to the non-doubt strategies, the politician can induce them to play his favorite outcome *zeros*, by strategically sending his messages. For example, if the citizens play *zeros* in response to the politician's message, say, m', he can induce them to play *zeros* by simply sending the message m'. In response to the doubt strategies, on the other hand, he cannot. However, under the structure of this game, in each period $t < T$, the citizens never play the doubt strategies in equilibrium. That is, they would play only the non-doubt strategies. The politician figures this out and hence knows that he can induce them to play *zeros*. He also knows that, under the information structure of the game, if they play *zeros*, they cannot figure out the actual state and thus cannot prove that he has sent untruthful messages. So, he can even continue to play the game without a significant disadvantage from sending misleading information. Consequently, the politician can successfully manipulate the citizens' actions in his favor in each period $t < T$, which is the result in Theorem 1.

To see why, under the structure of this game, the citizens never play the doubt strategies in equilibrium, suppose by way of contradiction that there exists an equilibrium in which they play doubt strategies. That is, in this equilibrium, they play only positive actions regardless of his messages. In response to these strategies, if the politician sends untruthful messages, the citizens will find it out because of the information structure of the game, which ensures that they can figure out the actual state if and only if they play positive actions. Hence, the politician will no longer play the game from the next period on, and he can get only his single period payoff in the current period t and cannot expect any future payoffs from period $t+1$ on. Since the citizens play positive actions, say, $a' > (0,0)$ and we have $a(0) = (0,0)$, where $a(0)$ is the citizens' optimal actions in the state $\theta = 0$, the politician's payoff in this case, $u(a')$, will be strictly less than $u(a(0))$ according to Assumption 1, which states that he strictly prefers the citizens' lower actions to their higher actions.

On the contrary, if the politician sends truthful messages, he can continue to play the game in the next period $t+1$. Hence, under the payoff structure of this game, he can get his single period payoff in the current period t and he can expect a discounted sum of his future expected payoffs from the next period $t+1$ on. Note that each citizen has a unique optimal action in a given state θ and her optimal action is increasing in θ, any actions $a > a(1)$ are dominated by $a(1)$ because the actions $a(1)$ are the citizens' optimal actions in the state $\theta = 1$ and $\theta = 1$ is the highest state. Hence, in equilibrium, the citizens play only the actions $a'' \leq a(1)$, which then means that the politician's single period payoff will be at least $u(a(1))$ (according to Assumption 1). In addition, according to Lemma 5 in the Appendix, a discounted sum of the politician's future expected payoffs from period $t+1$ to T is no less than $\delta \cdot u(a([0,1]))$. Consequently, by sending truthful messages, the politician's total expected payoff from period t to T will be at least $u(a(1)) + \delta u(a([0,1]))$.

Therefore, according to the condition in Theorem 1, $u(a(1)) + \delta u(a([0,1])) \geq u(a(0))$, the politician can get a higher payoff by sending truthful messages than by sending untruthful messages. Hence, he will send only truthful messages in response to the citizens' doubt strategies. These truthful messages, then, reveal the information about the actual states. Accordingly, by observing the messages, the citizens can figure out the actual state. In particular, when they observe a message $m \in [0, \theta_0]$, they can be certain that the actual state is within $[0, \theta_0]$. Note that, when the actual state is within $[0, \theta_0]$, the citizens' optimal actions are only *zeros*. Hence, in response to a message $m \in [0, \theta_0]$, the citizens have an incentive to play *zeros*, which clearly means that they have an incentive to play non-doubt strategies. That is, they have an incentive to deviate from their doubt strategies. As a consequence, the citizens never play the doubt strategies in equilibrium.

To sum up, when choosing between the doubt strategies and the non-doubt strategies, the citizens never choose the doubt strategies in equilibrium because of the structure of the game. Under this structure, the citizens' doubt strategies trigger the politician's reaction, which is for him to send truthful messages only. The politician's truthful messages reveal the information about the actual state. Hence, the citizens have an incentive to play according to his messages, which then means that they have an incentive to deviate from their doubt strategies. As a result, they play only the non-doubt strategies in equilibrium. In response to the non-doubt strategies, the politician can induce them to play *zeros*, by strategically sending his messages. Once the citizens play *zeros*, they cannot find out that he has sent untruthful messages. Hence, he can even continue to play the game without a significant disadvantage from sending misleading information. Therefore, the structure of the game enables the politician to successfully manipulate the actions of the rational citizens in his own favor, even against their own interests.

Theorem 1 is stable in that its result can remain unchanged even when the politician has imperfect information about the actual state, as modeled in Benabou and Laroque (1992) and Corneo (2006). Specifically, suppose that the politician can observe only a noisy signal $\theta + \varepsilon$ where θ is the actual state and ε is a random variable that is distributed on the interval $[-r, r]$ for some positive real r. Suppose further that an incumbent politician will be replaced with a new one if and only if the incumbent politician has sent a message $m \notin [\theta - r, \theta + r]$ and the citizens do not play *zeros*. The latter defines a tolerance level within which incorrect messages are forgiven.

In this model with the imperfectly informed politician, suppose that the random variables θ and ε are uniformly distributed. Then, whenever we have $r/2 < \theta_0$, Theorem 1 remains unchanged. This is because in this case, when the politician partially reveals the information (by sending misleading information only within the tolerance level), in response to the message $m \in [0, \theta_0 - r/2]$, the optimal actions of the citizens are still *zeros*. Note that the sufficient condition in Theorem 1 embodies the politician's incentive to send truthful messages in response to the citizens' doubt strategies, and it does not depend on the size of the state θ_0. Hence, this condition must remain valid even when we replace θ_0 with $\theta_0 - r/2$, and under this condition, the politician can still induce the citizens to play *zeros* by using the message $m \in [0, \theta_0 - r/2]$, just like he can do it by using the message $m \in [0, \theta_0]$ in the original model with the perfectly informed politician. In general, regardless of the distributions of θ and ε, if we have $2r < \theta_0$, Theorem 1 remains unchanged in the model with the imperfectly informed politician.

Next, Proposition 2 below shows that, in the second last period, that is, $t = T - 1$, the sufficient condition in Theorem 1, which is $u(a(1)) + \delta u(a([0,1])) \geq u(a(0))$, is also the necessary condition for the unique equilibrium outcome *zeros*. Therefore, Proposition 2 proves that the condition in Theorem 1 is indeed the weakest sufficient condition out of all the sufficient conditions in a game defined in Section 2.[11]

Proposition 2 *Suppose $\infty > T \geq 2$. In period $t = T - 1$, there exists only one equilibrium outcome and it is zeros if and only if the inequality $u(a(1)) + \delta u(a([0,1])) \geq u(a(0))$ holds.*

Finally, to substantiate Theorem 1 and Proposition 2, we consider the following simple example.

Example 1 *Suppose that the distribution of the states is uniform. Suppose further that we have $u(a) = H - \sqrt{a_1 a_2}$, $0 < \theta_0 < 0.5$, and $H > 0.5 - \theta_0$. The conditions $0 < \theta_0 < 0.5$ and $H > 0.5 - \theta_0$ guarantee $a([0,1]) > (0,0)$ and $u(a([0,1])) > 0$, respectively. In a given state θ, each citizen i's optimal action is $a_i(\theta) = \max\{0, \theta - \theta_0\}$. In addition, her optimal action throughout the whole states will be $a_i([0,1]) = 0.5 - \theta_0$. Thus, we obtain that $u(a(1)) = H - (1 - \theta_0)$, $u(a(0)) = H$, and $u(a([0,1])) = H - (0.5 - \theta_0)$. Therefore, the weakest sufficient condition (for the unique equilibrium outcome zeros whenever $t < T$) becomes $H \geq (1 + \frac{1}{\delta})(1 - \theta_0) - \frac{1}{2}$. In particular, when $\infty > T \geq 2$, in period $t = T - 1$, this is the necessary and sufficient condition.*

3.2 Necessary condition

Theorem 2 presents our second main result, the necessary condition for the structural manipulation.

Theorem 2 *In period $t < T$, if zeros are the unique equilibrium outcome, then the game satisfies the inequality $u(a(1)) + \delta u(a([0,1])) > u(a([0,1]))$.*

Remark 2 *The inequality $u(a(1)) + \delta u(a([0,1])) > u(a([0,1]))$ captures the situation in which the politician has a relatively high future payoff compared to his current period payoff. Accordingly, Theorem 2 means that if, in period t except the last period T, the citizens play only zeros in equilibrium, then the politician must have a relatively high future payoff. As*

[11] Logically, every sufficient condition must imply any arbitrary necessary condition. Since this condition is a necessary condition when $t = T - 1$, every sufficient condition in a game must imply this condition. Note that, in logic, when a condition Φ implies another condition Ψ, then we say that the condition Ψ is weaker than the condition Φ. Therefore, the condition in Theorem 1 is indeed the weakest sufficient condition.

such, by contraposition, if his future payoff is not relatively high, which situation is captured by the following inequality $u(a(1)) + \delta u(a([0,1])) \leq u(a([0,1]))$*, then the citizens can play positive actions in equilibrium even in period* $t < T$. *Therefore, Theorem 2 presents the condition under which the citizens can play the game in their own favor and hence can improve their payoffs, which in turn sheds some light on their (structural manipulation) problem.*

Proposition 3 below shows that, if there exist infinite periods and the discount factor δ is *one*, the necessary condition in Theorem 2, $u(a(1)) + \delta u(a([0,1])) > u(a([0,1]))$, is also the sufficient condition for the structural manipulation. Therefore, Proposition 3 proves that this condition is the strongest necessary condition out of all the necessary conditions in a game defined in Section 2.[12]

Proposition 3 *When* $T = \infty$ *and* $\delta = 1$*, there exists the unique equilibrium outcome, zeros, if and only if* $u(a(1)) + \delta u(a([0,1])) > u(a([0,1]))$ *holds.*

The strongest necessary condition $u(a(1)) + \delta u(a([0,1])) > u(a([0,1]))$ embodies the politician's incentive to send truthful messages in response to the citizens' doubt strategies. Compared to the weakest sufficient condition $u(a(1)) + \delta u(a([0,1])) \geq u(a(0))$ in Theorem 1, this condition represents a relatively small incentive. This relatively small incentive, however, can be sufficient for him to send truthful messages in response to the doubt strategies if there are infinite periods left and his discount factor δ is *one*. This is because he can benefit more from sending truthful messages in this setting of $T = \infty$ and $\delta = 1$.

Finally, to illustrate the results in this Subsection, we revisit the simple example in Subsection 3.1.

Example 2 *Consider Example 1 again. That is, the distribution of the states is uniform, and the politician' payoff is defined as* $u(a) = H - \sqrt{a_1 a_2}$. *We assume that* $0 < \theta_0 < 0.5$ *and* $H \geq 0.5 - \theta_0$. *Then, the strongest necessary condition for the unique equilibrium outcome, zeros, whenever* $t < T$ *becomes* $H > \frac{1}{2\delta} + \frac{1}{2} - \theta_0$. *Moreover, if we have* $T = \infty$ *and* $\delta = 1$*, the necessary and sufficient condition for the unique equilibrium outcome, zeros, becomes* $H > 1 - \theta_0$.

4 Conclusion

- Summary

[12] Since the condition $u(a(1)) + \delta u(a([0,1])) > u(a([0,1]))$ is a sufficient condition when $T = \infty$ and $\delta = 1$, every necessary condition in a game defined in Section 2 must be implied by this condition. Note that, in logic, when a condition Ψ is implied by another condition Φ, then we say that the condition Φ is stronger than the condition Ψ. Therefore, the condition in Theorem 2 is indeed the strongest necessary condition.

History has repeatedly shown that an informed politician can manipulate uninformed citizens in his or her favor. We have proposed the theory which shows that those citizens could be perfectly rational. The theory explains this possibility based on the game structure, related to both the payoff and the information. Under the payoff structure, the citizens choose their actions according to the politician's messages. Under the information structure, they might not be able to find out whether or not the politician sends misleading information. As a consequence, the politician (he) sends misleading information, and, without a significant disadvantage from it, he can successfully manipulate the "rational" citizens in his favor, even against their own interests. This phenomenon is the structural manipulation.

In this paper, we have first presented the weakest sufficient condition for the structural manipulation. This condition ensures that, if the politician has a sufficiently high future payoff, then, in every period t except the last period T, the citizens will play only the politician's favorite outcome in equilibrium. Hence, this result evidences that the politician can manipulate the rational citizens in his favor. We have next presented the strongest necessary condition for the structural manipulation. This condition ensures that, if the politician does not have a relatively high future payoff, then the citizens can play the game in their own favor, even in period $t < T$. Therefore, this result sheds some light on the structural manipulation problem.

- Discussion

How can the citizens solve this structural manipulation problem? The main cause of the problem, according to the theory, is the game structure. Hence, a natural approach to solving this problem would be to change the game structure itself. Since the game structure consists of two parts, there are two specific approaches. First, by changing the payoff structure, the citizens can solve the problem. In this approach, they can refer to Theorem 2 because it, by way of contraposition, presents a sufficient condition which ensures that they can play the game in their own favor. Second, the citizens can change the information structure so that they can figure out the actual state regardless of their taken actions. Under this (changed) information structure, once the politician sends misleading information, the citizens can always find it out and hence can punish him properly. Therefore, this certainty of the punishment can deter the politician from sending misleading information, and as a consequence, the citizens can solve their structural manipulation problem.

It should be emphasized that timing really matters in both approaches above. The citizens must change the game structure before they start the game. Otherwise, in the middle of the game, they might realize that they cannot help but follow the same logic as in the structural manipulation, and thus have no choice but to play the politician's favorite, even against their own interests. In addition, we would like to point out that the result in this paper is derived under the assumption that the citizens are rational. Hence, they fully understand the rules of the game and, while playing, they do their best to increase their own payoffs. Therefore, the citizens cannot improve their situations by examining the game more carefully or by making more effort not to play the politician's favorite.

- Final remarks

It is commonly said that history repeats itself. Painful history is no exception. Sometimes, it could be because we do not remember the history itself, but, often, it could be because we do not know what we really need to learn from the history. When it comes to the history related to the structural manipulation, just remembering chronological records and firmly deciding not to repeat the painful history would not guarantee that we will not repeat it in the future. We should never say that we will never repeat the painful history. Instead, **we should find the real causes of the effects in the history and, most of all, if we find any, we should take proper actions before we actually face a similar situation**.

5 Appendix: Proofs

1. Proposition 1

Proof. We show that there exists an equilibrium in which the politician sends the messages only out of the set $[0, \theta_0]$ and each citizen i always plays only one action $a_i = 0$.

Suppose that, in each period t, the politician chooses a message $m_t \in [0, \theta_0]$ and sends it to the citizens regardless of the actual state. Suppose further that the citizens play $a = (0, 0)$ in response to the messages in $[0, \theta_0]$ and play $a' > 0$ in response to the messages in $(\theta_0, 1]$ such that the actions a' maximize their expected payoffs throughout the whole states $[0, 1]$, that is, a'_i is defined as $\inf\{a_i \in \mathbb{R}_+ : \int_{[0,1]} \frac{\partial[\theta^2 - \{\theta - (\theta_0 + a_i)\}^2]}{\partial a_i} d\mu = 2\int_{[0,1]} \theta d\mu - 2(\theta_0 + a_i) \leq 0\}$, which is well-defined.

First, take into account the politician's incentive. When the citizens play the game according to the strategies designated above, by sending a message $m \in [0, \theta_0]$, the politician will get his current period payoff $u(0, 0)$ in a given period t. Note that $u(0, 0)$ is the highest current period payoff that he can get, according to Assumption 1. In addition, if the citizens play $(0, 0)$, they cannot figure out the actual state, and hence the politician can continue to play the game even when he sends untruthful messages. Therefore, sending only the messages in $[0, \theta_0]$ can be optimal in response to the citizens' strategies.

Next, take into account citizen i's incentive. In response to a message $m \in [0, \theta_0]$, since (citizen j plays) $a_j = 0$ and $w(0) = 0$, citizen i will get

$$w(a_j)\nu(\theta, a_i) = 0 \cdot [\theta^2 - \{\theta - (\theta_0 + a_i)\}^2] = 0$$

regardless of a_i. Hence, citizen i cannot improve her payoff by changing her action, which means that her action $a_i = 0$ is one of the best responses. In response to a message $m' \in (\theta_0, 1]$, her action a'_i can also be one of the best responses because the politician does not actually send any messages in $(\theta_0, 1]$. As a result, citizen i has no incentive to deviate from her strategy. Likewise, citizen j has no incentive to deviate from her strategy.

Therefore, the players' strategies are an equilibrium. ■

2. Theorem 1

The proof of Theorem 1 uses the following six Lemmas. Lemmas 1 to 5 show that the politician's expected payoff in each period is no less than $u(a([0,1]))$. Hence, these Lemmas guarantee that a discounted sum of the politician's future expected payoffs is no less than $\delta u(a([0,1]))$. Next, Lemma 6 shows that, if a discounted sum of the politician's future expected payoffs is δU^{t+1}, a sufficient condition for the unique equilibrium outcome *zeros* will be $u(a(1)) + \delta U^{t+1} \geq u(a(0))$. That is, Lemma 6 presents a conditional sufficient condition. Then, the sufficient condition for the unique equilibrium outcome *zeros*, $u(a(1)) + \delta u(a([0,1])) \geq u(a(0))$, directly follows from Lemmas 5 and 6.

Lemma 1 first shows that, if the game lasts only for finite periods, in the last period, the citizens play either $a([0,1])$ or $a(0)$ in equilibrium. Hence, the politician's expected payoff in the last period is either $u(a([0,1]))$ or $u(a(0))$. Since $u(a(0)) \geq u(a([0,1]))$, Lemma 1 guarantees that the politician's expected payoff in the last period is no less than $u(a([0,1]))$.

Lemma 1 *Suppose that T is finite. In the last period, the citizens play either $a([0,1])$ or $a(0)$.*

Proof. The proof of this lemma consists of three assertions as follows.

Assertion 1: The actions $a(0) = (0,0)$ can be an equilibrium outcome in the last period.

Assertion 2: In the last period, if citizen i plays a positive action a_i^* (> 0) with positive probability, then she will play this action a_i^* with probability one and the other citizen will also play this action $a_j^* = a_i^*$ with probability one. That is, whenever citizen i plays a positive action with positive probability, both the citizens will play the same action in every state.

Assertion 3: In the last period, if the citizens play positive actions a^* (> 0) with probability one, we always obtain $a^* = a([0,1])$.

To prove the Assertion 1, consider the strategy profile under which the politician chooses a message $m_t \in [0, \theta_0]$ and sends it to the citizens regardless of the actual state, and the citizens play $a = (0,0)$ in response to the messages in $[0, \theta_0]$ and play $a' > 0$ in response to the messages in $(\theta_0, 1]$ such that a'_i is $\inf\{a_i \in \mathbb{R}_+ : \int_{[0,1]} \frac{\partial \nu(\theta, a_i)}{\partial a_i} d\mu = 2\int_{[0,1]} \theta d\mu - 2(\theta_0 + a_i) \leq 0\}$. As shown in the proof of Proposition 1, this strategy profile can be an equilibrium, and in this equilibrium, the citizens play $a(0) = (0,0)$ with probability one in the last period.

To prove the Assertion 2, by way of contradiction, suppose that there exists an equilibrium in which, in the last period, citizen i plays a set of actions A'_i ($\neq \varnothing$) with positive probability and plays another set of actions A''_i ($\neq \varnothing$) with positive probability such that $A'_i \cap A''_i = \varnothing$ and either $\inf A'_i > 0$ or $\inf A''_i > 0$ or both. Without loss of generality, assume that there exist both a subset $\check{A}'_i \subset A'_i$ and a subset $\check{A}''_i \subset A''_i$ such that *i*) $\sup \check{A}'_i < \inf \check{A}''_i$ (that is, any action in $\check{A}'_i$ is less than every action in $\check{A}''_i$) and *ii*) citizen i plays both $\check{A}'_i$ and $\check{A}''_i$ with positive probabilities in this equilibrium. Suppose further that citizen i plays $\check{A}'_i$ in response to the politician's messages M' ($\neq \varnothing$), and she plays $\check{A}''_i$ in response to his messages M'' ($\neq \varnothing$). Since the setting of the citizens is symmetric, citizen j ($\neq i$) also plays $\check{A}'_j = \check{A}'_i$ in response to the politician's messages M' and plays $\check{A}''_j = \check{A}''_i$ in response to his messages M''.

Note that since $\partial^2\nu/\partial a_i^2 < 0$ and $\partial^2\nu/\partial\theta\partial a_i > 0$, given a set of states Θ, the optimal actions $a(\Theta)$ are uniquely determined, which in turn means that the same messages induce the same actions. Hence, by contraposition, different actions must be induced by different messages. Since $\check{A}'_i \cap \check{A}''_i = \varnothing$, we have $M' \cap M'' = \varnothing$. Since $\sup \check{A}'_i < \inf \check{A}''_i$, which means $u((\sup \check{A}'_1, \sup \check{A}'_2)) > u((\inf \check{A}''_1, \inf \check{A}''_2))$ according to Assumption 1, the politician can have higher payoffs by sending messages in M' than the payoffs by sending messages in M'', which then means that the politician has an incentive not to send any messages in M''. That is, he will not send the messages in M'' with positive probability. Note that citizen i plays $\check{A}''_i$ only in response to the messages in M''. Since the politician will not send the messages in M'' with positive probability, citizen i will not play $\check{A}''_i$ with positive probability. This contradiction completes the proof of the second assertion.

Finally, we will prove the Assertion 3. Consider an arbitrary equilibrium in which the citizen play actions $a^* = (a_1^*, a_2^*)$ such that $a_1^*, a_2^* > 0$ in every state $\theta \in [0,1]$. Since the action a_i^* is an equilibrium action in every state, a_i^* must maximize citizen i's expected payoff throughout the states in $[0,1]$. That is, a_i^* is $\inf\{a_i \in \mathbb{R}_+ : \int_{[0,1]} \frac{\partial\nu(\theta,a_i)}{\partial a_i} d\mu = 2\int_{[0,1]} \theta d\mu - 2(\theta_0 + a_i) \leq 0\}$, which is $a_i([0,1])$ by definition. Note that, since the setting is symmetric, citizen j also plays $a_j^* = a_j([0,1])$. Therefore, we obtain $a^* = a([0,1])$. ■

Suppose that there exists an equilibrium in which, in period t, the citizens play positive actions with positive probability. That is, in this equilibrium, the citizens do not play *zeros* with probability one in period t. Define a set $\bar{\Theta}^t$ as the smallest set of states such that in period t, in a state $\theta' \in [0,1] \setminus \bar{\Theta}^t$, the politician reveals the state θ' truthfully, or equivalently, in $\theta' \in [0,1] \setminus \bar{\Theta}^t$, he sends the truthful message $m' = \theta'$ and in another state $\theta \neq \theta'$, he does not send this message m'. Then, in this equilibrium, when the citizens observe this message m' ($= \theta' \in [0,1] \setminus \bar{\Theta}^t$), they are certain that the actual state is θ' ($= m'$) since the politician sends this message m' only in the state θ'.

Lemmas 2 to 4 analyze the citizens' behavior in the states $\bar{\Theta}^t$. Lemma 2 first shows that we have $\mu(\bar{\Theta}^t) > 0$ and that the citizens play only one action profile throughout the states

$\bar{\Theta}^t$. Next, Lemma 3 shows that the unique action profile, which the citizens play throughout $\bar{\Theta}^t$, is $a(\bar{\Theta}^t)$. Lastly, Lemma 4 shows that $a(\bar{\Theta}^t) \leq a([0,1])$, which then means $u(a(\bar{\Theta}^t)) \geq u(a([0,1]))$ according to Assumption 1. Therefore, these three Lemmas ensure that, in a state $\theta \in \bar{\Theta}^t$, the politician's expected payoff is at least $u(a([0,1]))$.

Lemma 2 *Consider an equilibrium in which the citizens play positive actions with positive probability in period t. Let a set $\bar{\Theta}^t$ denote the smallest set of states in period t such that, in a state $\theta' \in [0,1] \setminus \bar{\Theta}^t$, the politician reveals the state θ' truthfully. In this equilibrium, we have $\mu(\bar{\Theta}^t) > 0$, and, in a state $\theta \in \bar{\Theta}^t$, the citizens play only one action profile.*

Proof. To prove the first assertion, by way of contradiction, suppose that there exists an equilibrium such that $\mu(\bar{\Theta}^t) = 0$, that is, $\mu([0,1] \setminus \bar{\Theta}^t) = 1$. Since the politician sends a truthful message $m' = \theta'$ in a state $\theta' \in [0,1] \setminus \bar{\Theta}^t$, the equation $\mu([0,1] \setminus \bar{\Theta}^t) = 1$ means that the politician sends a truthful message in every state. Then, in response to a message $m'' \in [0, \theta_0]$, the citizens will play $a(0)$ and, in response to a message $m''' \in (\theta_0, 1]$, they will play $a(m''')$ ($> a(0)$). Hence, by sending a message $m'' \in [0, \theta_0]$, the politician can get $u(a(0))$ when $t = T$ and can get $u(a(0)) + \delta U^{t+1}$ when $t < T$, where we denote by δU^{t+1} the discounted sum of the politician's expected payoffs from period $t+1$ to T. By sending a message $m''' \in (\theta_0, 1]$, the politician can get $u(a(m''))$ when $t = T$ and can get $u(a(m''')) + \delta U^{t+1}$ when $t < T$. Since $u(a(0)) > u(a(m'''))$, the politician has an incentive to replace his message m''' with m'' in the states of $(\theta_0, 1]$, which shows that the politician has an incentive to deviate from his designated strategy in this equilibrium. Therefore, the hypothesis, $\mu(\bar{\Theta}^t) = 0$, leads to contradiction, which then means that we have $\mu(\bar{\Theta}^t) > 0$.

To prove the second assertion, by way of contradiction, suppose that citizen i plays two distinct nonempty sets of actions A_i'' and A_i''' with positive probabilities in the states $\bar{\Theta}^t$. We will show that the set $\bar{\Theta}^t$ includes a non-empty set of states Θ''' in which the politician reveals the states truthfully, which then contradicts the definition of $\bar{\Theta}^t$. Without loss of generality, assume that there exist both a subset $\breve{A}_i'' \subset A_i''$ and a subset $\breve{A}_i''' \subset A_i'''$ such that 1) $\sup \breve{A}_i'' < \inf \breve{A}_i'''$ and 2) citizen i plays both $\breve{A}_i''$ and $\breve{A}_i'''$ with positive probabilities in this equilibrium. Suppose that citizen i plays $\breve{A}_i''$ in response to the politician's messages M'' ($\neq \varnothing$), and she plays $\breve{A}_i'''$ in response to his messages M''' ($\neq \varnothing$). Since $\partial^2 \nu / \partial a_i^2 < 0$ and $\partial^2 \nu / \partial \theta \partial a_i > 0$, different actions are induced by different messages, as explained in the proof of Lemma 1. Hence, we must have $M'' \cap M''' = \varnothing$. Since the setting of the citizens is symmetric, citizen j ($\neq i$) also plays $\breve{A}_j'' = \breve{A}_i''$ in response to the politician's messages M'' and plays $\breve{A}_j''' = \breve{A}_i'''$ in response to his messages M'''.

In response to the citizens' strategies, in every state θ, if the politician sends a message $m''' \in M'''$ such that $m''' \neq \theta$ (that is, m''' is an untruthful message), he can get at most $u((\inf \breve{A}_1'', \inf \breve{A}_2''))$ and, if he sends a message $m'' \in M''$, he can get at least $u((\sup \breve{A}_1', \sup \breve{A}_2'))$.

Since $\sup \check{A}_i'' < \inf \check{A}_i'''$, the politician can have a higher payoff by sending a message $m'' \in M''$ than that by sending a message $m''' \in M'''$ if the actual state θ is not m'''. Hence, the politician will not send a message $m''' \in M'''$ in a state $\theta \neq m'''$. This, by contraposition, means that the politician sends a message $m''' \in M'''$ only if an actual state θ is indeed m'''. Then, when the citizens observe the politician's message $m''' \in M'''$, they are certain that the actual state θ is m'''. Note that the citizens play $\check{A}_i'''$ with positive probability and they play it in some states of $\bar{\Theta}^t$. Since the actions $\check{A}_i'''$ are induced by only the messages in M''', the politician sends the messages in M''' in some states of $\bar{\Theta}^t$. Hence, there are a nonempty set of states Θ''' $(\subset \bar{\Theta}^t)$ in which the politician reveals the states truthfully, by sending truthful messages in M'''. As a consequence, we have $\Theta''' \neq \varnothing$ and $\Theta''' \subset \bar{\Theta}^t$. In addition, in a state $\theta''' \in \Theta'''$, the politician reveals the state θ''' truthfully. This contradicts the definition of the set $\bar{\Theta}^t$, which then completes the proof. ■

Lemma 3 *In every state $\theta \in \bar{\Theta}^t$, the citizens play $a(\bar{\Theta}^t)$.*

Proof. The result directly follows from Lemma 2. ■

Lemma 4 *We have $a(\bar{\Theta}^t) \leq a([0,1])$.*

Proof. By way of contradiction, suppose that there exists an equilibrium in which we have $a_i(\bar{\Theta}^t) > a_i([0,1])$. In this equilibrium, we will first show that there exists a state θ' $\in [0,1] \setminus \bar{\Theta}^t$ in which the citizens play $a(\theta')$ such that $a(\bar{\Theta}^t) > a(\theta')$, and will next show that this hypothesis, $a_i(\bar{\Theta}^t) > a_i([0,1])$, leads to contradiction.

Since $a_i(\bar{\Theta}^t) > a_i([0,1]) \geq 0$, we have that $\partial \int_{\bar{\Theta}^t} \nu(\theta, a_i(\bar{\Theta}^t)) d\mu / \partial a_i = 0$ by definition of $a_i(\bar{\Theta}^t)$ and that $\partial \int_{[0,1]} \nu(\theta, a_i([0,1])) d\mu / \partial a_i \leq 0$ by definition of $a_i([0,1])$. Since $\partial^2 \nu / \partial a_i^2 < 0$ and $a_i(\bar{\Theta}^t) > a_i([0,1])$, we have that $\int_{[0,1]} \frac{\partial \nu(\theta, a_i(\bar{\Theta}^t))}{\partial a_i} d\mu < 0$. Since $\int_{[0,1]} \frac{\partial \nu(\theta, a_i(\bar{\Theta}^t))}{\partial a_i} d\mu = \int_{\bar{\Theta}^t} \frac{\partial \nu(\theta, a_i(\bar{\Theta}^t))}{\partial a_i} d\mu + \int_{[0,1] \setminus \bar{\Theta}^t} \frac{\partial \nu(\theta, a_i(\bar{\Theta}^t))}{\partial a_i} d\mu$, we obtain that

$$\int_{[0,1] \setminus \bar{\Theta}^t} \frac{\partial \nu(\theta, a_i(\bar{\Theta}^t))}{\partial a_i} d\mu < 0. \tag{1}$$

This inequality (1) implies that there exists a subset $\Theta' \subset [0,1] \setminus \bar{\Theta}^t$ such that $\mu(\Theta') > 0$ and $\partial \nu(\theta', a_i(\bar{\Theta}^t)) / \partial a_i < 0$ for every state $\theta' \in \Theta'$. Since $a_i(\bar{\Theta}^t) > 0$, $\partial \nu(\theta', a_i(\bar{\Theta}^t)) / \partial a_i < 0$, and $\partial^2 \nu / \partial a_i^2 < 0$, we obtain that

$$a_i(\bar{\Theta}^t) > a_i(\theta')$$

for every state $\theta' \in \Theta'$. Since the setting of the citizens is symmetric, we have that

$$a_j(\bar{\Theta}^t) \ (= a_i(\bar{\Theta}^t)) > a_j(\theta') \ (= a_i(\theta'))$$

for every state $\theta' \in \Theta'$. Note that, in every state of $[0,1] \setminus \bar{\Theta}^t$, the politician sends truthful messages and the citizens can figure out the actual states. Consequently, since $\Theta' \subset [0,1] \setminus \bar{\Theta}^t$ and $\mu(\Theta') > 0$, there exists a state $\theta' \in \Theta'$ in which the citizens play $a(\theta')$ such that $a(\bar{\Theta}^t) > a(\theta')$.

By its definition, the set $\bar{\Theta}^t$ is the smallest set of states such that, in a state $\theta'' \in [0,1] \setminus \bar{\Theta}^t$, the politician sends a truthful message $m'' = \theta''$ and, in another state θ ($\neq \theta''$), he does not send this message m''. This implies that there exists a state $\theta''' \in \bar{\Theta}^t$ such that in this state θ''', the politician sends untruthful messages $m \neq \theta'''$. However, since $a_i(\bar{\Theta}^t) > a_i(\theta')$, which means $u(a(\theta')) > u(a(\bar{\Theta}^t))$, in the state θ''', he can get the higher payoff $u(a(\theta'))$ by sending the message $m' = \theta'$ than by sending the messages designated in the equilibrium, which induces the citizens to play $a(\bar{\Theta}^t)$ according to Lemma 3. Hence, in this state θ''', he has an incentive to deviate from the equilibrium. This contradiction completes the proof. ■

Lemma 5 aggregates Lemmas 1 to 4 and proves that, given any period $t \leq T$, a discounted sum of the politician's expected payoffs from period t to T is no less than $u(a([0,1]))$.

Lemma 5 *Given an equilibrium, in each period t, we have $U^t \geq u(a([0,1]))$ where we denote by U^t the discounted sum of the politician's expected payoffs from period t to T.*

Proof. When $t = T$, we have $U^T \in \{u(a([0,1])), u(a(0))\}$ according to Lemma 1. Since $u(a(0)) \geq u(a([0,1]))$, we obtain $U^T \geq u(a([0,1]))$.

Suppose $t < T$. In this period t, consider the case in which the citizens play $a(0)$ in response to some message m. Then, the politician can get $u(a(0))$ in every state by sending the message m regardless of the actual state. Note that $u(a(0))$ is the highest payoff that the politician can get in this period and, if the citizens play $a(0) = (0,0)$, the politician can continue to play the game even when he sends untruthful messages. Hence, the politician will send the message m, which induces the actions $a(0)$, and as a result, the citizens will play $a(0)$. Since $u(a(0)) \geq u(a([0,1]))$, in this case, we obtain $U^t \geq u(a([0,1]))$.

Next, we only need to consider the case in which the citizens do not play $a(0)$ in response to any message from the politician. Suppose a set of states $\bar{\Theta}^t$ is defined as in Lemma 2. That is, the set $\bar{\Theta}^t$ denotes the smallest set of states such that in period t, in a state $\theta' \in [0,1] \setminus \bar{\Theta}^t$, the politician reveals the state truthfully. Then, according to Lemmas 2 and 3, we obtain

$$U^t \geq \mu(\bar{\Theta}^t) \cdot u(a(\bar{\Theta}^t)) + \int_{[0,1] \setminus \bar{\Theta}^t} u(a(\theta)) d\mu + \mu([0,1] \setminus \bar{\Theta}^t) \delta U^{t+1} \tag{2}$$

where we denote by δU^{t+1} the discounted sum of the politician's expected payoffs from period $t+1$ to T. In the right side of the inequality (2), the first term, $\mu(\bar{\Theta}^t) \cdot u(a(\bar{\Theta}^t))$, denotes the politician's single period expected payoff in the states $\bar{\Theta}^t$. The other two terms,

$\int_{[0,1]\setminus\bar{\Theta}^t} u(a(\theta))d\mu + \mu([0,1]\setminus\bar{\Theta}^t)\delta U^{t+1}$, denote the discounted sum of the politician's expected payoffs from period t to T in the states $[0,1]\setminus\bar{\Theta}^t$. The second term, $\int_{[0,1]\setminus\bar{\Theta}^t} u(a(\theta))d\mu$, denotes his single period expected payoff when he sends truthful messages in $[0,1]\setminus\bar{\Theta}^t$. When the politician sends truthful messages, he can continue to play in the next period. Thus, the third term, $\mu([0,1]\setminus\bar{\Theta}^t)\delta U^{t+1}$, denotes his discounted expected payoff from $t+1$ to T.

Note that, in the states $\bar{\Theta}^t$, the politician might send truthful messages while the citizens cannot figure out the actual states. For example, suppose that the politician sends a message $m' = \theta'$ in two distinct states θ', $\theta'' \in \bar{\Theta}^t$. In this case, when the citizens observe the message m', they cannot figure out the actual states. However, in the state θ', the politician sends the truthful message $m' = \theta'$. Hence, he will continue to play in the next period. Therefore, the probability that the politician continues to play in the next period is no less than $\mu([0,1]\setminus\bar{\Theta}^t)$. This is why we need an inequality sign "$\geq$" in the inequality (2).

It suffices to show that 1) $u(a(\bar{\Theta}^t)) \geq u(a([0,1]))$ and 2) $\int_{[0,1]\setminus\bar{\Theta}^t} u(a(\theta))d\mu + \mu([0,1]\setminus\bar{\Theta}^t)\delta U^{t+1} \geq \mu([0,1]\setminus\bar{\Theta}^t)u(a([0,1]))$. The assertion "1)" follows from Lemma 4. To prove the assertion "2)," we need to take into account the politician's incentive in this equilibrium. In every state $\theta' \in [0,1]\setminus\bar{\Theta}^t$, the politician gets $u(a(\theta')) + \delta U^{t+1}$ by sending the truthful message $m' = \theta'$. The politician, however, can get at least $u(a(\bar{\Theta}^t))$ by sending untruthful messages. Since the politician sends the truthful message $m' = \theta'$ in the state θ', we must have $u(a(\theta')) + \delta U^{t+1} \geq u(a(\bar{\Theta}^t))$. Therefore, according to Lemma 4, we obtain $u(a(\theta')) + \delta U^{t+1} \geq u(a([0,1]))$ for every state $\theta' \in [0,1]\setminus\Theta$, which proves the assertion 2). Therefore, from the inequality (2), we obtain $U^t \geq u(a([0,1]))$. ■

Lemma 6 proves that, if the current period t is not the last period, that is, $t < T$, given a discounted sum of the politician's future expected payoffs δU^{t+1}, a sufficient condition for the unique equilibrium outcome *zeros* is $u(a(1)) + \delta U^{t+1} \geq u(a(0))$.

Lemma 6 *In period $t < T$, if $u(a(1)) + \delta U^{t+1} \geq u(a(0))$, the actions $a(0)$ $(= (0,0))$ are the unique equilibrium outcome, where we denote by δU^{t+1} the discounted sum of the politician's expected payoffs from period $t+1$ to T.*

Proof. It is easy to see that the actions $a(0)$ can possibly be an equilibrium outcome in period t.

To prove the uniqueness, by way of contradiction, suppose that there exists an equilibrium in which citizen i plays actions in A'_i with positive probability such that $a_i(0) = 0 \notin A'_i$. Without loss of generality, assume $0 < \inf A'_i$. Suppose that citizen i plays A'_i in response to the politician's messages M'. Since the setting of the citizens is symmetric, citizen j $(\neq i)$ also plays $A'_j = A'_i$ in response to the politician's messages M'.

There are two possible cases: 1) the citizens play $a(0)$ in response to some message from the politician and 2) they do not play $a(0)$ in response to any message from the politician.

First, consider the case "1)" in which the citizens play $a(0)$ in response to some message. Let a message m^0 induce the actions $a(0)$. Then, by sending the message m^0, the politician will get $u(a(0)) + \delta U^{t+1}$, and, by sending the messages in M', the politician will get at most $u((\inf A'_i, \inf A'_j)) + \delta U^{t+1}$. Since $u(a(0)) > u((\inf A'_i, \inf A'_j))$, the politician will not send messages in M' and the citizens will not play A'_i with positive probability, which contradicts our hypothesis that the citizens play the actions A'_i with positive probability.

Second, consider the case "2)" in which the citizens do not play $a(0)$ in response to any message. Note that, since $\partial^2\nu/\partial a_i^2 < 0$ and $\partial^2\nu/\partial\theta\partial a_i > 0$, in equilibrium, citizen i plays actions only within $[a_i(0), a_i(1)]$. That is, in equilibrium, citizen i's lowest action is $a_i(0)$ and her highest action is $a_i(1)$. Hence, by sending a truthful message, the politician will get at least $u(a(1)) + \delta U^{t+1}$, and, by sending an untruthful message, he will get strictly less than $u(a(0))$ (since in the case 2), the citizens play positive actions only). Since $u(a(1)) + \delta U^{t+1} \geq u(a(0))$, the politician has an incentive to send a truthful message in every state. That is, when the actual state is $\theta' \in [0, 1]$, the politician will send a truthful message $m' = \theta'$, and he will not send this message m' in another state θ ($\neq \theta'$). Then, by observing the politician's messages, the citizens can figure out the actual states. Hence, when they observe the messages in $[0, \theta_0]$, they are certain that the actual state is within $[0, \theta_0]$. Note that, when the actual state is within $[0, \theta_0]$, the citizens' optimal actions are $a(0)$. Consequently, in response to the messages in $[0, \theta_0]$, the citizens have an incentive to deviate from the equilibrium and will play $a(0)$. This contradiction completes the proof. ■

Finally, Theorem 1 is an immediate consequence of Lemmas 5 and 6.

Proof of Theorem 1. The result, the sufficient condition, directly follows from Lemmas 5 and 6. ■

3. Proposition 2

Proof. Since Theorem 1 proves sufficiency in this Proposition, we need to show necessity only. By way of contraposition, suppose that we have $u(a(1)) + \delta u(a([0, 1])) < u(a(0))$. It suffices to show that there exists an equilibrium whose outcome is not *zeros* in the second last period $T - 1$.

There are two possible cases: 1) $u(a(1)) + \delta u(a([0, 1])) \leq u(a([0, 1]))$ and 2) $u(a(1)) + \delta u(a([0, 1])) > u(a([0, 1]))$. In the case "1)," according to Theorem 2, there exists an equilibrium whose outcome is not *zeros*.

Suppose the case "2)" (that is, $u(a(1)) + \delta u(a([0, 1])) > u(a([0, 1]))$). Then, we have

$$u(a([0, 1])) < u(a(1)) + \delta u(a([0, 1])) < u(a(0)). \tag{3}$$

Note that, since the functions $u(a)$ and $a([0, \theta])$ are continuous in a and θ, respectively, the function $u(a([0, \theta]))$ is continuous in θ. In addition, we have $a(0) = a([0, \theta_0])$ since for any

$\theta \in [0, \theta_0]$, we have $\partial\nu(\theta, 0)/\partial a_i = 2\{\theta - (\theta_0 + 0)\} \leq 0$. Hence, from the inequality (3), there exists a state $\theta^* \in (\theta_0, 1)$ such that $u(a(1)) + \delta u(a([0, 1])) = u(a([0, \theta^*]))$.

Consider the following strategy profile. In each period $t < T - 1$, the politician chooses a message $m_t \in [0, \theta_0]$ and sends it to the citizens regardless of the actual state, and the citizens play $a = (0, 0)$ in response to the messages in $[0, \theta_0]$ and play $a' > 0$ in response to the messages in $(\theta_0, 1]$ such that a'_i is $\inf\{a_i \in \mathbb{R}_+ : \int_{[0,1]} \frac{\partial\nu(\theta, a_i)}{\partial a_i} d\mu = 2\int_{[0,1]} \theta d\mu - 2(\theta_0 + a_i) \leq 0\}$. In the second last period, $T - 1$, the politician sends truthful messages in every state of $(\theta^*, 1]$, and he sends untruthful messages in almost every state of $[0, \theta^*]$ by sending only one message $m = 0$ throughout these states $[0, \theta^*]$. The citizens play $a([0, \theta^*])$ in response to the message $m = 0$, play $a(m')$ in response to a message $m' \in (\theta^*, 1]$, and play $a(1)$ in response to a message $m'' \notin (\theta^*, 1] \cup \{0\}$. In the last period T, the politician almost surely sends untruthful messages by sending only one message $m = 0$ throughout the states $[0, 1]$. The citizens play $a([0, 1])$ regardless of his messages. We will show that this strategy profile induces an equilibrium.

It is easy to see that, in each period $t < T-1$ and in the last period $t = T$, all the players, the politician and the citizens, have no incentive to deviate from this strategy profile. So, we focus on their incentives in the second last period.

First, take into account the politician's incentive in the second last period $T - 1$. Note that, for any $\theta' \in [0, 1]$, since $\partial^2\nu/\partial\theta\partial a_i > 0$, we have $\partial\nu(a_i, \theta')/\partial a_i \geq \partial\nu(a_i, \theta)/\partial a_i$ for any $\theta \leq \theta'$, and thus we obtain

$$a(\theta') \equiv \inf\{a_i \in \mathbb{R}_+ : \partial\nu(a_i, \theta')/\partial a_i\} \leq 0\}$$
$$\geq \inf\{a_i \in \mathbb{R}_+ : \int_{[0,\theta']} \frac{\partial\nu(a_i, \theta)}{\partial a_i} d\mu\} \leq 0\} \equiv a([0, \theta']).$$

Hence, under this strategy profile, the actions $a([0, \theta^*])$ are the lowest actions that the citizens play in period $T-1$. Note that, in the last period T, under this strategy profile, the politician's expected payoff is $u(a([0, 1]))$.

In response to the citizens' strategies, in a state $\theta' \in (\theta^*, 1]$, by sending a truthful message, the politician can get $u(a(\theta')) + \delta u(a([0, 1]))$. By sending untruthful messages, he can get at most $u(a([0, \theta^*]))$ since $a([0, \theta^*])$ are the lowest actions that the citizens play in this period. Since $\theta' \leq 1$, which means $u(a(\theta')) \geq u(a(1))$, we obtain $u(a(\theta')) + \delta u(a([0, 1])) \geq u(a(1)) + \delta u(a([0, 1])) = u(a([0, \theta^*]))$. Accordingly, the politician has no incentive to deviate from his strategy, and hence will send the truthful message in a state $\theta' \in (\theta^*, 1]$. In a state $\theta'' \notin [0, \theta^*] \setminus \{0\}$, by sending a truthful message, he will get $u(a(1)) + \delta u(a([0, 1]))$ and, by sending the untruthful message $m = 0$, he can get $u(a([0, \theta^*]))$. Since $u(a(1)) + \delta u(a([0, 1])) = u(a([0, \theta^*]))$, sending the message $m = 0$ is still one of his best responses, which shows that he has no incentive to deviate from his strategy. Obviously, in the state $\theta = 0$, he had better send the message $m = 0$. As a result, the politician has no incentive to deviate from his strategy in every state in the second last period $T - 1$.

Next, take into account the citizens' incentives in the second last period $T - 1$. In response to a message $m' \in (\theta^*, 1]$, their actions $a(m')$ are the best responses because the politician sends a truthful message and the actual state is indeed m'. In response to the message $m = 0$, their actions $a([0, \theta^*])$ are also the best responses because he sends the message $m = 0$ throughout the states $[0, \theta^*]$ and $a([0, \theta^*])$ maximize their expected payoffs throughout the states $[0, \theta^*]$ by definition. Finally, in response to a message $m'' \notin (0, \theta^*]$, their actions $a(1)$ can be one of the best responses because the politician does not actually send any messages in $(0, \theta^*]$. As a result, the citizens have no incentive to deviate from their strategies in the second last period $T - 1$.

Consequently, the suggested strategy profile induces an equilibrium, and this equilibrium has non-*zero* outcomes in $t = T - 1$. The existence of this equilibrium completes the proof. ■

4. Theorem 2

Proof. By way of contraposition, suppose that $u(a(1)) + \delta u(a([0, 1])) \leq u(a([0, 1]))$. It suffices to show that there exists an equilibrium whose outcome is not *zeros*. Consider the following strategy profile. The politician sends only one message $m' \in [0, 1]$ regardless of the states. The citizens play $a([0, 1])$ in response to the message m' and play $a(1)$ in response to a message $m \in [0, 1] \setminus \{m'\}$. We will show that this strategy profile can be an equilibrium.

First, take into account the politician's incentive.

In each period $t < T$, in response to the citizens' strategies, by sending the message m', the politician can get at least $u(a([0, 1]))$. By sending a message $m \neq m'$, he can get at most $u(a(1)) + \delta U^{t+1}$ where we denote by δU^{t+1} the discounted sum of his expected payoffs from period $t + 1$ to T.

We will show that $\delta U^{t+1} \leq \delta u(a([0, 1]))$. Note that, when the players play the game according to the strategy profile designated in this proof, in each period the politician gets $u(a([0, 1]))$, and in the next period he will be discharged from his position (since he sends the untruthful message m'). Hence, if he follows the designated strategy in period $t + 1$, the discounted sum of his payoffs from $t + 1$ to T will be $\delta U^{t+1} = \delta\{u(a([0, 1])) + \delta \cdot 0\}$ $= \delta u(a([0, 1]))$. If he sends a truthful message in period $t + 1$ and will follow the designated strategy in period $t + 2$, the discounted sum of his payoffs will be $\delta U^{t+1} = \delta\{u(a(1)) + \delta \cdot u(a([0, 1]))\} \leq \delta u(a([0, 1]))$ since $u(a(1)) + \delta u(a([0, 1])) \leq u(a([0, 1]))$. Similarly, we can show that in any cases, the discounted sum of his expected payoffs from period $t + 1$ to T will be less than $\delta u(a([0, 1]))$.

Therefore, in each period $t < T$, the politician can get $u(a([0, 1]))$ by following the strategy, and can get at most $u(a(1)) + \delta u(a([0,1]))$ by deviating from the strategy, which shows that he has no incentive to deviate from his strategy since $u(a(1)) + \delta u(a([0,1])) \leq u(a([0,1]))$.

It is easy to see that, in the last period $t = T$, the politician still has no incentive to deviate from his strategy.

Next, take into account the citizens' incentives.

In response to the message m', the citizens' actions $a([0,1])$ are the best responses. This is because the politician sends only one message m' throughout all the states $[0,1]$ and the actions $a([0,1])$ maximize the citizens' expected payoffs throughout the states $[0,1]$ by definition, that is, $a_i([0,1])$ is defined as $\inf\{a_i \in \mathbb{R}_+ : \int_{[0,1]} \frac{\partial \nu(\theta, a_i)}{\partial a_i} d\mu = 2\int_{[0,1]} \theta d\mu - 2(\theta_0 + a_i) \leq 0\}$. Hence, they have an incentive to stick to their strategies. In response to a message $m \in [0,1] \setminus \{m'\}$, their actions $a(1)$ can be one of the best responses because the politician does not actually send any message $m \neq m'$. Consequently, the citizens have no incentive to deviate from their strategies in response to any messages from the politician.

Therefore, the strategy profile suggested in this proof is an equilibrium, and its unique outcome is $a([0,1])$, which is not *zeros*. ■

5. Proposition 3

The proof of Proposition 3 uses the following Lemmas 7, 8, and 9.

We first introduce a notation for simplicity. Given any positive real U^{t+1}, let $U(\Theta; U^{t+1})$ denote

$$U(\Theta; U^{t+1}) = \mu(\Theta) u(a(\Theta)) + \int_{[0,1]\setminus\Theta} u(a(\theta)) d\mu + \mu([0,1] \setminus \Theta) \cdot \delta U^{t+1}.$$

Then, $U(\Theta; U^{t+1})$ is well-defined and it can be viewed as the discounted sum of the politician's expected payoffs from period t to $T = \infty$, when the discounted sum of his expected payoffs from period $t+1$ to $T = \infty$ is δU^{t+1} and, in period t, he sends untruthful messages only in the states Θ, that is, he sends truthful messages in the states $[0,1] \setminus \Theta$.

Lemma 7 *Let U^{t+1} be a non-negative real. Suppose that a set of states $\Theta' \in \beta[0,1]$ and a state $\theta' \in [\theta_0, 1]$ satisfy the following three conditions: i) $a(\Theta') > 0$, ii) $u(a(\theta)) + \delta U^{t+1} \geq u(a(\Theta'))$ for every state $\theta \notin \Theta'$, and iii) $a([0,\theta']) \geq a(\Theta')$. Then we have $U(\Theta'; U^{t+1}) \geq U([0,\theta']; U^{t+1})$.*

Proof. According to the definition of the function $U(\cdot; U^{t+1})$, we have

$$U(\Theta'; U^{t+1}) = \mu(\Theta') u(a(\Theta')) + \int_{[0,1]\setminus\Theta'} u(a(\theta)) d\mu + \delta\mu([0,1] \setminus \Theta') \cdot U^{t+1}$$

and

$$U([0,\theta'];U^{t+1}) = \mu([0,\theta'])u(a([0,\theta'])) + \int_{[0,1]\setminus[0,\theta']} u(a(\theta))d\mu + \delta\mu([0,1]\setminus[0,\theta'])\cdot U^{t+1}.$$

Since $a([0,\theta']) \geq a(\Theta')$, we obtain $u(a([0,\theta'])) \leq u(a(\Theta'))$. Hence, we have $U(\Theta';U^{t+1}) - U([0,\theta'];U^{t+1})$

$$\begin{aligned}
&\geq \mu(\Theta')u(a(\Theta')) + \int_{[0,1]\setminus\Theta'} u(a(\theta))d\mu + \delta\mu([0,1]\setminus\Theta')\cdot U^{t+1}\\
&\quad -\mu([0,\theta'])u(a(\Theta')) - \int_{[0,1]\setminus[0,\theta']} u(a(\theta))d\mu - \delta\mu([0,1]\setminus[0,\theta'])\cdot U^{t+1}\\
&= \mu(\Theta'\setminus[0,\theta'])u(a(\Theta')) - \mu([0,\theta']\setminus\Theta')u(a(\Theta')) + \int_{[0,\theta']\setminus\Theta'} u(a(\theta))d\mu\\
&\quad - \int_{\Theta'\setminus[0,\theta']} u(a(\theta))d\mu + \delta\mu([0,\theta']\setminus\Theta')U^{t+1} - \delta\mu(\Theta'\setminus[0,\theta'])U^{t+1}\\
&= \int_{[0,\theta']\setminus\Theta'} \{u(a(\theta)) + \delta U^{t+1} - u(a(\Theta'))\}d\mu - \int_{\Theta'\setminus[0,\theta']} \{u(a(\theta)) + \delta U^{t+1} - u(a(\Theta'))\}d\mu.
\end{aligned}$$

We will show that the last part of the equalities above is non-negative.

If Θ' is almost surely the same as $[0,\theta']$, that is, $\mu(\Theta'\setminus[0,\theta']) = 0$ and $\mu([0,\theta']\setminus\Theta') = 0$, then, obviously we have $\int_{[0,\theta']\setminus\Theta'}\{u(a(\theta)) + \delta U^{t+1} - u(a(\Theta'))\}d\mu - \int_{\Theta'\setminus[0,\theta']}\{u(a(\theta)) + \delta U^{t+1} - u(a(\Theta'))\}d\mu = 0$.

Suppose that Θ' is not almost surely the same as $[0,\theta']$. Then, we have either $\mu(\Theta'\setminus[0,\theta']) > 0$ or $\mu([0,\theta']\setminus\Theta') > 0$ or both. By way of contradiction, if we had $\mu(\Theta'\setminus[0,\theta']) > 0$ and $\mu([0,\theta']\setminus\Theta') = 0$, then, since $a(\theta) > a([0,\theta']) > 0$ for every $\theta \in \Theta'\setminus[0,\theta']$, we would obtain $a(\Theta') > a([0,\theta'])$, which contradicts the hypothesis *iii*) in this Lemma, $a([0,\theta']) \geq a(\Theta')$. Therefore, we need to consider only two cases: 1) $\mu(\Theta'\setminus[0,\theta']) = 0$ and $\mu([0,\theta']\setminus\Theta') > 0$ and 2) $\mu(\Theta'\setminus[0,\theta']) > 0$ and $\mu([0,\theta']\setminus\Theta') > 0$.

First, consider the case "1)" in which we have both $\mu(\Theta'\setminus[0,\theta']) = 0$ and $\mu([0,\theta']\setminus\Theta') > 0$. Note that we have $u(a(\theta)) + \delta U^{t+1} - u(a(\Theta')) \geq 0$ for every $\theta \notin \Theta'$ according to the hypothesis *ii*) in this Lemma. Hence, we obtain

$$\int_{[0,\theta']\setminus\Theta'} \{u(a(\theta)) + \delta U^{t+1} - u(a(\Theta'))\}d\mu \geq 0 = \int_{\Theta'\setminus[0,\theta']} \{u(a(\theta)) + \delta U^{t+1} - u(a(\Theta'))\}d\mu.$$

Next, consider the other case "2)" in which we have both $\mu(\Theta'\setminus[0,\theta']) > 0$ and $\mu([0,\theta']\setminus\Theta') > 0$. Note that the hypotheses *i*) and *iii*) in this Lemma mean that $a([0,\theta']) \geq a(\Theta') > 0$, and thus we obtain $a(\theta') > 0$. Then, for any state $\theta''' \in \Theta'\setminus[0,\theta']$, since $\theta''' > \theta'$, we have $a(\theta''') > a(\theta') > 0$. In addition, for any state $\theta'' \in [0,\theta']\setminus\Theta'$ and for any state $\theta''' \in \Theta'\setminus[0,\theta']$, since $\theta'' \leq \theta' < \theta'''$, we have $a(\theta'') < a(\theta''')$. This last inequality means $u(a(\theta'')) > u(a(\theta'''))$ according to Assumption 1. Thus, we must have

$$u(a(\theta'')) + \delta U^{t+1} - u(a(\Theta')) > u(a(\theta''')) + \delta U^{t+1} - u(a(\Theta')). \tag{4}$$

Note that since $a([0,\theta']) \geq \ a(\Theta') > 0$, for $i \in \{1,2\}$, we have

$$\begin{aligned} & a_i([0,\theta']) \geq a_i(\Theta') \equiv a_i^* > 0 \\ \iff & \int_{[0,\theta']} \frac{\partial \nu(\theta, a_i^*)}{\partial a_i} d\mu \geq \int_{[0,\theta']} \frac{\partial \nu(\theta, a_i([0,\theta']))}{\partial a_i} d\mu = 0 = \int_{\Theta'} \frac{\partial \nu(\theta, a_i^*)}{\partial a_i} d\mu \\ \iff & \int_{[0,\theta']\setminus\Theta'} \frac{\partial \nu(\theta, a_i^*)}{\partial a_i} d\mu \geq \int_{\Theta'\setminus[0,\theta']} \frac{\partial \nu(\theta, a_i^*)}{\partial a_i} d\mu. \end{aligned}$$

In addition, since $\theta'' \leq \theta' < \theta'''$, we obtain $\partial\nu(\theta'', a_i^*)/\partial a_i = 2\{\theta'' - (\theta_0 + a_i^*)\} < 2\{\theta''' - (\theta_0 + a_i^*)\} = \partial\nu(\theta''', a_i^*)/\partial a_i$. Finally, since $\int_{[0,\theta']} \frac{\partial \nu(\theta, a_i([0,\theta']))}{\partial a_i} d\mu = 0$ and $\partial^2\nu/\partial\theta\partial a_i > 0$, we obtain $\partial\nu(\theta', a_i([0,\theta']))/\partial a_i \geq 0$, which then means $\partial\nu(\theta', a_i^*)/\partial a_i \geq 0$ since $\partial^2\nu/\partial a_i^2 < 0$, and thus we obtain $\partial\nu(\theta''', a_i^*)/\partial a_i > 0$ since $\partial^2\nu/\partial\theta\partial a_i > 0$.

Consequently, for any state $\theta'' \in [0,\theta'] \setminus \Theta'$ and for any state $\theta''' \in \Theta' \setminus [0,\theta']$, we have $\partial\nu(\theta'', a_i^*)/\partial a_i < \partial\nu(\theta''', a_i^*)/\partial a_i$ and $\partial\nu(\theta''', a_i^*)/\partial a_i \ > 0$, and as a result, the inequality $\int_{[0,\theta']\setminus\Theta'} \frac{\partial \nu(\theta, a_i^*)}{\partial a_i} d\mu \geq \int_{\Theta'\setminus[0,\theta']} \frac{\partial \nu(\theta, a_i^*)}{\partial a_i} d\mu$ means that we must have

$$\mu([0,\theta'] \setminus \Theta') > \mu(\Theta' \setminus [0,\theta']). \tag{5}$$

Therefore, from the inequalities (4) and (5), we obtain

$$\int_{[0,\theta']\setminus\Theta'} \{u(a(\theta)) + \delta U^{t+1} - u(a(\Theta'))\} d\mu > \int_{\Theta'\setminus[0,\theta']} \{u(a(\theta)) + \delta U^{t+1} - u(a(\Theta'))\} d\mu.$$

This last inequality completes the proof. ■

We introduce a function for simplicity. For any set of states $\Theta \in \beta[0,1]$ such that $[0,\theta_0] \subset \Theta$, define a function $U^t(\Theta)$ as

$$U^t(\Theta) = \frac{1}{1 - \delta(1 - \mu(\Theta))} \left[\mu(\Theta) u(a(\Theta)) + \int_{[0,1]\setminus\Theta} u(a(\theta)) d\mu \right].$$

Then, $U^t(\Theta)$ is well-defined as the discounted sum of the politician's expected payoffs from period t to $T = \infty$, when the politician sends untruthful messages only in the states Θ, that is, he sends truthful messages in the states $[0,1] \setminus \Theta$. For example, $U^t([0,1]) = u(a([0,1]))$.

Lemma 8 *Let $T = \infty$. Suppose that, if $u(a(1)) > 0$, then, for any $\theta \in [\theta_0, 1]$ such that*

$u(a(1)) + \delta u(a([0,1])) \leq u(a([0,\theta]))$, we have $\delta\{U^{t+1}([0,\theta]) - U^{t+1}([0,1])\} \geq u(a([0,\theta])) - u(a([0,1]))$. Then, there exists only one equilibrium outcome and it is zeros if and only if we have $u(a(1)) + \delta u(a([0,1])) > u(a([0,1]))$.

Proof. Since Theorem 2 proves necessity in this Lemma, we need to show sufficiency only. That is, when $u(a(1)) + \delta u(a([0,1])) > u(a([0,1]))$, we will show that there exists the unique equilibrium outcome, *zeros*.

Suppose $u(a(1)) + \delta u(a([0,1])) > u(a([0,1]))$. There are two possible cases: 1) $u(a(1)) + \delta u(a([0,1])) \geq u(a(0))$ and 2) $u(a(1)) + \delta u(a([0,1])) < u(a(0))$. In the case "1)," according to Theorem 1, there exists only one equilibrium outcome in each period, and it is *zeros*.

Suppose the case "2)" (that is, $u(a(1)) + \delta u(a([0,1])) < u(a(0))$). Then, we have $u(a(0)) > u(a(1)) + \delta u(a([0,1])) > u(a([0,1]))$. Note that, since $u(a([0,1])) \geq 0$, we have

$$\begin{aligned} &u(a(1)) + \delta u(a([0,1])) > u(a([0,1])) \\ \iff &u(a(1)) > (1-\delta)u(a([0,1])) \geq 0, \end{aligned}$$

and thus, according to the hypothesis in this Lemma, we have $\delta\{U^{t+1}([0,\theta]) - U^{t+1}([0,1])\} \geq u(a([0,\theta])) - u(a([0,1]))$ for any $\theta \in [\theta_0, 1]$ such that $u(a(1)) + \delta u(a([0,1])) \leq u(a([0,\theta]))$.

By way of contradiction, suppose that there exists an equilibrium whose outcome is not *zeros*. In this equilibrium, let a set T' denote the set of periods in which the citizens play positive actions with positive probability. In each period $t' \in T'$, define a set of states $\bar{\Theta}^{t'}$ as in Lemma 2, that is, the set $\bar{\Theta}^{t'}$ is the smallest set of states such that, in a state $\theta' \in [0,1]\backslash\bar{\Theta}^{t'}$, the politician reveals the state truthfully. Let a class $\{\bar{\Theta}^{t'}\}_{t' \in T'}$ be the class of such sets $\bar{\Theta}^{t'}$ in each period $t' \in T'$.

According to Lemma 3, in each period $t' \in T'$, the citizens play $a(\bar{\Theta}^{t'})$ throughout the states $\bar{\Theta}^{t'}$. According to Lemma 4, we have $a(\bar{\Theta}^{t'}) \leq a([0,1])$, which means $u(a(\bar{\Theta}^{t}))$ $\geq u(a([0,1]))$ by Assumption 1. Let a real $\check{u}$ be $\inf\{u(a(\bar{\Theta}^{t'})) : \bar{\Theta}^{t'} \in \{\bar{\Theta}^{t'}\}\}$. Then, we have $u(a([0,1])) \leq \check{u}$. Note that $u(a([0,\theta]))$ is continuous in θ. In addition, note that we have $u(a(0)) = u(a([0,\theta_0]))$ and $\check{u} \leq u(a(0))$. Hence, since $u(a([0,1])) \leq \check{u} \leq u(a([0,\theta_0]))$, there exists a state $\check{\theta} \in [\theta_0, 1]$ such that $u(a([0,\check{\theta}])) = \check{u}$. Consequently, in each period $t' \in T'$, we have $u(a(\bar{\Theta}^{t'})) \geq \check{u} = u(a([0,\check{\theta}]))$ and $a([0,\check{\theta}]) \geq a(\bar{\Theta}^{t'})$.

Now, we will show that $u(a(1)) + \delta u(a([0,1])) \leq u(a([0,\check{\theta}]))$. By way of contradiction, if we had $u(a(1)) + \delta u(a([0,1])) > u(a([0,\check{\theta}]))$, there should exist period t'' such that $u(a(1)) + \delta u(a([0,1])) > u(a(\bar{\Theta}^{t''}))$ because $u(a([0,\check{\theta}]))$ is the infimum of $\{u(a(\bar{\Theta}^{t'})) : \bar{\Theta}^{t'} \in \{\bar{\Theta}^{t'}\}\}$. According to Lemma 5, we have $\delta U^{t''+1} \geq \delta u(a([0,1]))$ where we denote by $\delta U^{t''+1}$ the discounted sum of the politician's expected payoffs from period $t''+1$ on. Hence, in period t'', we would have $u(a(1)) + \delta U^{t''+1} > u(a(\bar{\Theta}^{t''}))$. Then, in some states of $\bar{\Theta}^{t''}$, by sending a truthful message, the politician could get at least $u(a(1)) + \delta U^{t''+1}$. By following his strategy designated in this equilibrium, which induces the citizens' actions $a(\bar{\Theta}^{t''})$ according to Lemma 3, he could get $u(a(\bar{\Theta}^{t''}))$. Since $u(a(1)) + \delta U^{t''+1} > u(a(\bar{\Theta}^{t''}))$, he would have an incentive to deviate from the equilibrium, which proves that we must have $u(a(1)) + \delta u(a([0,1])) \leq u(a([0,\check{\theta}]))$.

Then, according to the hypothesis in this Lemma, we have $\delta\{U^{t+1}([0,\check{\theta}]) - U^{t+1}([0,1])\} \geq u(a([0,\check{\theta}])) - u(a([0,1]))$. Since $U^{t+1}([0,1]) = u(a([0,1]))$, we obtain

$$\begin{aligned}
&u(a(1)) + \delta u(a([0,1])) > u(a([0,1])) \\
\Longleftrightarrow\ &u(a(1)) + \delta U^{t+1}([0,1]) > u(a([0,1])) \\
\Longrightarrow\ &u(a(1)) + \delta U^{t+1}([0,1]) + \delta\{U^{t+1}([0,\check{\theta}]) - U^{t+1}([0,1])\} \\
&> u(a([0,1])) + \{u(a([0,\check{\theta}])) - u(a([0,1]))\} \\
\Longleftrightarrow\ &u(a(1)) + \delta U^{t+1}([0,\check{\theta}]) > u(a([0,\check{\theta}])).
\end{aligned}$$

Note that, in the inequality $u(a(1)) + \delta U^{t+1}([0,\check{\theta}]) > u(a([0,\check{\theta}]))$, since $T = \infty$, the value $U^{t+1}([0,\check{\theta}])$ does not depend on period t, that is, $U^{t+1}([0,\check{\theta}]) = U^{t''+1}([0,\check{\theta}])$ for any period t''. Hence, we have $u(a(1)) + \delta U^{t+1}([0,\check{\theta}]) > u(a([0,\check{\theta}]))$ for every t. In addition, since $u(a([0,\check{\theta}]))$ is the infimum of $\{u(a(\bar{\Theta}^{t'}))\}_{t' \in T'}$, there exists a period $\check{t} \in T'$ such that

$$u(a(1)) + \delta U^{\check{t}+1}([0,\check{\theta}]) > u(a(\bar{\Theta}^{\check{t}})). \tag{6}$$

Moreover, note that, for each period $t' \in T'$, we have $a([0,\check{\theta}]) \geq a(\bar{\Theta}^{t'})$ and, in each period $t'' \notin T'$, the citizens play only $a(0)$, which is strictly less than $a([0,\check{\theta}])$. Hence, according to Lemmas 3 and 7, we have

$$\delta U^{t+1} \geq \delta U^{t+1}([0,\check{\theta}]) \text{ for every period } t \tag{7}$$

where we denote by δU^{t+1} the discounted sum of the politician's expected payoffs from period $t+1$ to $T = \infty$ in this equilibrium. Therefore, from the inequalities (6) and (7), we obtain

$$u(a(1)) + \delta U^{\check{t}+1} > u(a(\bar{\Theta}^{\check{t}})) \tag{8}$$

in period $\check{t}$ where we denote by $\delta U^{\check{t}+1}$ the discounted sum of the politician's expected payoffs from period $\check{t}+1$ to ∞ in this equilibrium.

The inequality (8), however, leads to contradiction. In some states of $\bar{\Theta}^{\check{t}}$, by sending a truthful message, the politician can get at least $u(a(1)) + \delta U^{\check{t}+1}$. By following his strategy designated in this equilibrium, which induces the citizens' actions $a(\bar{\Theta}^{\check{t}})$ according to Lemma

3, he can get $u(a(\bar{\Theta}^{\check{t}}))$. Since $u(a(1)) + \delta U^{\check{t}+1} > u(a(\bar{\Theta}^{\check{t}}))$, he has an incentive to deviate from the equilibrium. This contradiction shows that there exists only one equilibrium outcome and it is *zeros*, which then completes the proof. ■

Lemma 9 *Let $T = \infty$. There exists a positive real $\delta^* \in (0, 1)$ such that, for any discount factor $\delta \geq \delta^*$, there exists only one equilibrium outcome and it is zeros if and only if we have $u(a(1)) + \delta u(a([0, 1])) > u(a([0, 1]))$.*

Proof. According to Lemma 8, it suffices to show that there exists a positive real $\delta^* \in (0, 1)$ such that, given any discount factor $\delta \geq \delta^*$, the condition of the hypothesis in Lemma 8 holds good. That is, given any $\delta \geq \delta^*$, if $u(a(1)) > 0$, we need to show $\delta\{U^{t+1}([0, \theta]) - U^{t+1}([0, 1])\} \geq u(a([0, \theta])) - u(a([0, 1]))$ for any $\theta \in [\theta_0, 1]$ such that $u(a(1)) + \delta u(a([0, 1])) \leq u(a([0, \theta]))$. Note that, if we have $\delta^*\{U^{t+1}([0, \theta]) - U^{t+1}([0, 1])\} \geq u(a([0, \theta])) - u(a([0, 1]))$ for any θ such that $u(a(1)) + \delta^* u(a([0, 1])) \leq u(a([0, \theta]))$, then for every $\delta \geq \delta^*$, we also have $\delta\{U^{t+1}([0, \theta]) - U^{t+1}([0, 1])\} \geq u(a([0, \theta])) - u(a([0, 1]))$ for any θ such that $u(a(1)) + \delta u(a([0, 1])) \leq u(a([0, \theta]))$. Hence, we only need to show that there exists a discount factor $\delta^* \in (0, 1)$ such that, given the discount factor δ^*, the condition of the hypothesis in Lemma 8 holds good.

Suppose that $u(a(1)) > 0$. We will show that there exists such a discount factor $\delta^* \in (0, 1)$.

Since $u(a(1)) > 0$ and $u(a(0)) > \max\{u(a([0, 1])), u(a(1))\}$, there exists $\delta_1 \in (0, 1)$ such that $u(a(0)) > u(a(1)) + \delta_1 u(a([0, 1])) > u(a([0, 1]))$. By using this discount factor δ_1, define a state θ_1 as $\sup\{\theta \in [\theta_0, 1] : u(a(1)) + \delta_1 u(a([0, 1])) \leq u(a([0, \theta_1]))\}$. This state θ_1 is well-defined and, since $u(a([0, \theta]))$ is (weakly) decreasing and continuous in θ, we have $\theta_1 \in (\theta_0, 1)$ and $u(a(1)) + \delta_1 u(a([0, 1])) = u(a([0, \theta_1]))$. We will show that there exists a discount factor $\delta_2 \in (0, 1)$ such that, given this discount factor δ_2, the condition of the hypothesis in Lemma 8 holds good for any state $\theta \in [\theta_0, \theta_1]$. That is, given δ_2, we will show that $\delta_2\{U^{t+1}([0, \theta]) - U^{t+1}([0, 1])\} \geq u(a([0, \theta])) - u(a([0, 1]))$ for any $\theta \in [\theta_0, 1]$ such that $u(a(1)) + \delta_1 u(a([0, 1])) \leq u(a([0, \theta]))$. Then, if we choose $\delta^* = \max\{\delta_1, \delta_2\}$, this discount factor δ^* satisfies all the conditions.

Note that, given an arbitrary discount factor δ, since $\delta U^{t+1}([0, \theta]) = \frac{\delta}{1-\delta(1-\mu([0,\theta]))} \left[\mu([0, \theta])u(a([0, \theta])) + \int_{[0,1]\setminus[0,\theta]} u(a(\theta))d\mu\right]$ and $U^{t+1}([0, 1]) = u(a([0, 1]))$, we obtain

$$\begin{aligned}
&\delta\{U^{t+1}([0,\theta]) - U^{t+1}([0,1])\} - \{u(a([0,\theta])) - u(a([0,1]))\}\\
&= \delta U^{t+1}([0,\theta]) - u(a([0,\theta])) + (1-\delta)u(a([0,1]))\\
&= \frac{\delta\mu([0,\theta]) - 1 + \delta - \delta\mu([0,\theta])}{1-\delta(1-\mu([0,\theta]))}u(a([0,\theta]))\\
&+\frac{\delta}{1-\delta(1-\mu([0,\theta]))}\int_{[0,1]\setminus[0,\theta]} u(a(\theta))d\mu + (1-\delta)u(a([0,1]))\\
&= \frac{-(1-\delta)u(a([0,\theta]))}{1-\delta(1-\mu([0,\theta]))} + \frac{\delta\int_{[0,1]\setminus[0,\theta]} u(a(\theta))d\mu}{1-\delta(1-\mu([0,\theta]))} + (1-\delta)u(a([0,1])).
\end{aligned}$$

For any $\theta \in [\theta_0, 1]$ and for any $\delta \in (0,1)$, we obtain $\frac{-(1-\delta)}{1-\delta(1-\mu([0,\theta]))} < 0$, $u(a(0)) \geq u(a([0,\theta]))$, $\frac{\delta}{1-\delta(1-\mu([0,\theta]))} > 0$, and $u(a(\theta)) \geq u(a(1))$. Hence, we also obtain

$$\begin{aligned}
&\delta\{U^{t+1}([0,\theta]) - U^{t+1}([0,1])\} - \{u(a([0,\theta])) - u(a([0,1]))\}\\
&= \frac{-(1-\delta)u(a([0,\theta]))}{1-\delta(1-\mu([0,\theta]))} + \frac{\delta\int_{[0,1]\setminus[0,\theta]} u(a(\theta))d\mu}{1-\delta(1-\mu([0,\theta]))} + (1-\delta)u(a([0,1]))\\
&\geq \frac{-(1-\delta)u(a(0))}{1-\delta(1-\mu([0,\theta]))} + \frac{\delta\int_{[0,1]\setminus[0,\theta]} u(a(1))d\mu}{1-\delta(1-\mu([0,\theta]))} + (1-\delta)u(a([0,1]))\\
&= \frac{-(1-\delta)u(a(0))}{1-\delta(1-\mu([0,\theta]))} + \frac{\delta(1-\mu([0,\theta]))u(a(1))}{1-\delta(1-\mu([0,\theta]))} + (1-\delta)u(a([0,1])).
\end{aligned}$$

In addition, for any $\theta \in [\theta_0, \theta_1]$, we have $\frac{-(1-\delta)}{1-\delta(1-\mu([0,\theta]))} \geq \frac{-(1-\delta)}{1-\delta(1-\mu([0,\theta_0]))}$ since $\mu([0,\theta]) \geq \mu([0,\theta_0])$ and we have $\frac{\delta(1-\mu([0,\theta]))}{1-\delta(1-\mu([0,\theta]))} \geq \frac{\delta(1-\mu([0,\theta_1]))}{1-\delta(1-\mu([0,\theta_1]))}$ since $\mu([0,\theta]) \leq \mu([0,\theta_1])$. Hence, we obtain

$$\begin{aligned}
&\delta\{U^{t+1}([0,\theta]) - U^{t+1}([0,1])\} - \{u(a([0,\theta])) - u(a([0,1]))\}\\
&\geq \frac{-(1-\delta)u(a(0))}{1-\delta(1-\mu([0,\theta]))} + \frac{\delta(1-\mu([0,\theta]))u(a(1))}{1-\delta(1-\mu([0,\theta]))} + (1-\delta)u(a([0,1]))\\
&\geq \frac{-(1-\delta)u(a(0))}{1-\delta(1-\mu([0,\theta_0]))} + \frac{\delta(1-\mu([0,\theta_1]))u(a(1))}{1-\delta(1-\mu([0,\theta_1]))} + (1-\delta)u(a([0,1])).
\end{aligned}$$

Since $\lim_{\delta\longrightarrow 1} \frac{-(1-\delta)}{1-\delta(1-\mu([0,\theta_0]))} = 0$, $\lim_{\delta\longrightarrow 1} \frac{\delta(1-\mu([0,\theta_1]))}{1-\delta(1-\mu([0,\theta_1]))} = \frac{1-\mu([0,\theta_1])}{\mu([0,\theta_1])} > 0$, and $u(a(1)) > 0$, there exists a real $\delta_2 \in (0,1)$ such that

$$\frac{-(1-\delta_2)u(a(0))}{1-\delta_2(1-\mu([0,\theta_0]))} + \frac{\delta_2(1-\mu([0,\theta_1]))u(a(1))}{1-\delta_2(1-\mu([0,\theta_1]))} + (1-\delta_2)u(a([0,1])) \geq 0.$$

Finally, if $\delta^* = \max\{\delta_1, \delta_2\}$, then we have

$$\delta^*\{U^{t+1}([0,\theta]) - U^{t+1}([0,1])\} - \{u(a([0,\theta])) - u(a([0,1]))\} \geq 0$$

for any $\theta \in [\theta_0, 1]$ such that $u(a(1)) + \delta^* u(a([0,1])) \leq u(a([0,\theta]))$. Since δ_1, δ_2 $\in (0,1)$, we have $\delta^* \in (0,1)$ as well. The existence of this discount factor $\delta^* \in (0,1)$ completes the proof. ■

Proof of Proposition 3. The result directly follows from Lemma 9. ■

6 References

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