Screening without Single Crossing

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November 18, 2022

Job Market Paper
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Abstract

This paper provides a sufficient condition under which a general screening problem can be reduced, without the principal's payoff loss, to one with the single crossing property. The sufficient condition requires the agent's types to be ordered in a way that *two* marginal rates of substitution are both increasing in the type order. The monotonicity of one marginal rate of substitution allows the optimal mechanism to use only an allocation subset. The monotonicity of the other marginal rate of substitution ensures that the agent's preferences over the allocation subset satisfy the single crossing property. I apply the result to various economic problems with multi-dimensional allocations: multi-product monopoly, product line design, Bayesian persuasion, and delegation.

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1 Introduction

1.1 Motivation

A key to overcoming the ubiquitous problem of asymmetric information rests in the well-known "single crossing property" (Milgrom and Shannon, 1994). It says that in the principal-agent relationship with hidden information, the agent types and allocations can be ordered so that higher types have increasingly stronger preferences for higher allocations. Thus, if the screening problem satisfies the single crossing property, every implementable choice rule has to satisfy "type-monotonicity"; that is, the optimal choice from every menu of allocations is monotone in type. Also, a choice rule can be implemented if only local incentive compatibility constraints according to the type order are satisfied, which provides a simple representation of the total rent as a function of the choice rule. Since the type order is independent of the choice rule, the representation simplifies the screening problem into a maximization problem of only choice rule subject to the monotonicity constraint.

The single crossing property, however, is often a strong assumption. In particular, it requires the agent's preferences to be ordered by a single-dimensional characteristic, which is inapplicable to multi-product monopoly and numerous other screening problems. In absence of single crossing, the set of binding constraints depends on the choice rule, and this makes the optimal mechanism difficult to predict. The optimal mechanism may take many different forms depending on the agent's type distribution. For example, in the single-product monopoly, it is well-known that single price is optimal (Riley and Zeckhauser, 1983). However if the monopolist sells more than two goods, the optimal selling mechanism ranges from the simplest pure bundling (Pavlov, 2011; Haghpanah and Hartline, 2021) and mixed bundling (Manelli and Vincent, 2006) to probabilistic bundling and even a continuum of lotteries (Thanassoulis, 2004; Hart and Nisan, 2019), depending on the distribution of consumers.

This paper seeks to go beyond single crossing. I ask the following question. When, in general, can a screening problem be reduced to one that satisfies the single crossing property? I call a screening problem reducible to a single crossing problem, or *single-crossing-reducible*, under a subset of allocations X if (i) the principal can restrict attention to X without payoff loss when screening the agent and (ii) the agent's preferences over X satisfy the single crossing property. In such situations, the optimal mechanism can be found by solving a much more tractable problem.

I derive a sufficient condition for single-crossing-reducibility characterized by the agent's type set. The sufficient condition requires the agent types to be totally ordered in a way that *two* marginal rates of substitution are both increasing in the same type order. The monotonicity of one marginal rate of substitution ensures that the optimal mechanism can restrict attention to an allocation subset X. The monotonicity of the other marginal rate of substitution guarantees that the agent's preferences over X satisfy the single crossing property.

I apply this result to various economic settings to derive new sufficient conditions for the optimality of simple mechanisms as well as to generalize some previous sufficient conditions. Because all the sufficient conditions are only about the type set, the simple mechanisms are robustly optimal with respect to the distribution over the type set. First, in multiple good monopoly, I provide a sufficient condition for upgrade pricing, i.e., a menu of product bundles ordered by a set inclusion, which generalizes the optimality result of pure bundling in Haghpanah and Hartline (2021). The sufficient condition consists of *monotone valuation* and *monotone MRS*: Valuation for every good and MRS between each pair of the goods are monotone according to the same type order. Moreover, I show that if the valuation for each good is strictly monotone, monotone MRS is necessary and sufficient for the robust optimality of upgrade pricing with respect to the distribution over the type set.

In Bayesian persuasion, I extend the sufficient condition for the nested interval structure of acceptance sets in Guo and Shmaya (2019) to a larger class of payoff functions. In delegation, I characterize when no screening, i.e. offering the same cap regardless of the agent bias, is optimal. The corresponding sufficient condition is about the state distribution, and my result can also extend the Tanner (2018) condition about risk aversion to a more general form involving payoff functions.

The rest of the paper is organized as the follows. Section 2 introduces the model, followed by an illustrative example in Section 3. Section 4 provides a sufficient condition for the reducibility of screening problem to a single crossing problem. Section 5 uses the result to obtain sufficient conditions for the optimality of simple mechanisms that have nested forms. Section 6 discusses various applications in a unified setting and Section 7 concludes.

1.2 Literature Review

In the screening literature, the restrictiveness of the single crossing property is well-documented (Hörner, 2008) and its violation becomes inevitable especially when the agent types or allocations are multi-dimensional (Rochet and Stole, 2003). As explained in Rochet and Choné (1998), the violation makes it extremely difficult to generalize the analysis of the one-dimensional analogue. This has led the literature to focus its attention on the certification of the optimality of particular candidate mechanisms, and significant contributions have been made by using the duality approach along this direction (Rochet and Choné, 1998; Cai, Devanur, and Weinberg, 2016; Daskalakis, Deckelbaum, and Tzamos, 2017; Carroll, 2017; Haghpanah and Hartline, 2021).

A branch of the screening literature has attempted to relax the single crossing property while minimizing the loss in tractability.[1] In the standard setting of one dimensional type/allocation with money transfers, Araujo and Moreira (2010) and Schottmüller (2015) allow a one-time violation of single crossing property and characterize when the solution has U-shaped and monotone solutions, respectively. In multi-dimensional screening, Carroll (2012) studies when local incentive compatibility (IC) constraints imply global ones, which is a desirable property of the single crossing property. In the current paper, I offer a condition under which downward IC constraints combined with optimality are sufficient for full incentive constraints.

In dealing with screening problems without single crossing property, the role of downward IC constraints dates back to Moore (1984) and Matthews and Moore (1987). With two dimensional allocations with money transfers, Matthews and Moore (1987) characterizes optimal mechanisms that can be non-monotonic, which implies that non-local IC constraints might be essential, unlike problems with single crossing property. In their paper, the key step is finding a sufficient condition about the preferences where a solution of the relaxed problem with only downward IC constraints is that of the original problem, which requires utility functions of type for every pair of allocations to cross at most twice in a particular fashion. Finding a sufficient condition for the equivalence of the two solutions is also a critical step in our analysis, but our condition takes a different form for the reduction to a screening problem with single crossing property.

There is also a body of papers decomposing a multi-dimensional type set into one dimensional paths with single crossing property so that solving the screening problem for each path is sufficient for obtaining the solution for the original problem (Armstrong, 1996; Eső and Szentes, 2007; Haghpanah and Hartline, 2021; Bergemann, Bonatti, Haupt, and Smolin, 2021; Yang, 2021). Among the aforementioned papers, Haghpanah and Hartline (2021) and Yang (2021) are closest to this paper. Haghpanah and Hartline (2021) finds a sufficient condition for the reduction of multiple good monopoly to single good monopoly with grand bundle. The result in this paper allows us to not only recover their condition but also derive the optimality condition of more general selling mechanisms such as upgrade pricing (see Section 6.1). Yang (2021) studies the effectiveness of the use of a socially wasteful component in a specific setting in which money transfers are allowed and preferences are additively separable. The paper uses

[1]For a similar approach in signaling, see, for example, Chen, Ishida, and Suen (2022).

similar proof methods, but I study a more general screening problem to characterize when the problem can be reduced to a single crossing problem. Being specialized to the setting, the single-crossing-reducibility condition in this paper coincides with the optimality condition of Yang (2021) (see Remark 4).

2 Model

2.1 Setup

I first define a screening problem with a single principal and a single agent. A screening environment (Θ, Z, U) is defined by the agent's finite type space Θ, an allocation set Z, assumed to be a compact subset of a Euclidean space endowed with Lebesgue measure, and the principal and agent's utility functions $U_P(z,\theta), U_A(z,\theta) : Z \times \Theta \to \mathbb{R}$ that are continuous in $z \in Z$. Here, I do not allow any mixing between allocations, that is, only degenerate distributions over Z are allowed.[2] Also, note that we do not assume the quasilinearity or the independence of state about the players' preferences. Hence, it allows for every possible state dependence and complementarity between the components of elements of Z in the players' preferences. Lastly, for each type the agent's outside option value is normalized to be zero.

In the screening environment, the agent privately knows his type $\theta \in \Theta$ while the principal only has a prior belief $\mu \in \Delta\Theta$. The principal can screen the agent by using a mechanism.

2.2 Optimal IC mechanism

According to the revelation principle, I can restrict attention to direct mechanisms, functions of the agent's type, defined as $h : \Theta \to Z$. Throughout this text, a direct incentive compatible (IC) mechanism is a direct mechanism that satisfies both incentive compatibility (IC) and individual rationality (IR) constraints:

$$\begin{aligned} &IC: \ U_A(h(\theta),\theta) \geq U_A(h(\theta'),\theta) && \text{for all } \theta,\theta' \in \Theta \\ &IR: \ U_A(h(\theta),\theta) \geq 0 && \text{for all } \theta \in \Theta \end{aligned}$$

I denote the set of all the direct incentive compatible mechanisms as M. Then, the principal's problem is finding the *optimal IC mechanism under Z* which maximizes her own expected

[2]This assumption is with little loss of generality. For example, Z might be interpreted as the set of mixtures over a finite set of pure allocations.

utility among all the elements of M.

$$\max_{h \in M} \mathbb{E}[U_P(h(\theta), \theta)] \tag{1}$$

Next, I consider an alternative problem of the principal where the allocation set is restricted to be a compact subset X of Z. The setting is same except that the principal now has to choose every allocation from X for all type reports. Let M_X denote the set of direct incentive compatible mechanisms whose range is a subset of X. Then, the principal's problem can be written as the following by replacing M with M_X in (1):

$$\max_{h \in M_X} \mathbb{E}[U_P(h(\theta), \theta)] \tag{2}$$

I call the solution an *optimal IC mechanism under X*. The existence of both optimal IC mechanisms can be easily shown by the compactness argument.

2.3 Single crossing screening problem

Here, I define a class of screening problems with a desirable ordering property that makes them much more analytically tractable.

Definition 1 *A screening problem (Θ, X, U) is a **single crossing** problem if there exists a total order $\succeq_X$ on X and a total order $\succeq_\Theta$ on Θ s.t. for every $x' \succeq_X x$ and $\theta' \succeq_\Theta \theta$,*

$$\begin{aligned} U_A(x', \theta) \geq U_A(x, \theta) &\Rightarrow U_A(x', \theta') \geq U_A(x, \theta') \\ U_A(x', \theta) > U_A(x, \theta) &\Rightarrow U_A(x', \theta') > U_A(x, \theta') \\ U_A(x', \theta) \geq 0 &\Rightarrow U_A(x', \theta') \geq 0 \end{aligned} \tag{P}$$

The first two inequalities imply that higher type has a stronger preference for higher allocations. Also, the last inequality says that higher type has a stronger preference for each allocation to outside option of the type. As the term implies, in a single crossing screening problem the agent's preference over allocations has the *single crossing property* (SCP).

The single crossing condition of the screening problem (Θ, X, U) facilitates solving the problem by specifying the set of binding constraints. Because it preserves the preference for a better allocation over a worse one as type goes up, if every pair of local incentive constraints hold, all the IC constraints do.[3] Moreover, the condition also preserves the preference for each

[3] Given a total order $\succeq_\Theta$ over the type set Θ, local IC constraint means the IC constraint between type θ and θ' with $\theta' \succeq_\Theta \theta$ such that there does not exist another type between them, *i.e.*, $\theta' \succeq_\Theta \theta'' \succeq_\Theta \theta$.

allocation over the outside option as type goes up, and thus, only the IR constraint for the lowest type needs to be satisfied.

A well-known example of a single crossing screening problem is the following single good monopoly problem.

Example 1 *Suppose that (Θ, X, U) is a screening problem equivalent to the single good monopoly. That is, $\Theta \subseteq \mathbb{R}_+$ is the finite set of the consumer's possible values for the good, and $X = [0,1] \times \mathbb{R}_+$ is the allocation set where $(q,p) \in X$ means a lottery with probability q for obtaining the good and the price p. The payoff functions are defined by*

$$U_A((q,p),\theta) = q\theta - p \text{ and } U_P((q,p),\theta) = p.$$

Also, $(0,0)$ is the type independent outside option $\emptyset$. Then, the single crossing condition of (Θ, X, U) can be proven if we choose a total order on Θ as the standard order $\geq$ on $\mathbb{R}$ and a total order on $\succeq_X$ defined as

$$(q,p) \succeq_X (q',p') \Leftrightarrow q \geq q',$$

where $\emptyset = (0,0)$ is the worst allocation according to $\succeq_X$. More specifically, it can be shown that $U_A|_{X\times\Theta}$ satisfies the single crossing property with $(\succeq_X, \geq)$ because for $x = (q,p) \succeq_X (q',p') = x'$,

$$U_A(x,\theta) - U_A(x',\theta) = (q-q')\theta - (p-p') : \text{nondecreasing in } \theta.$$

■

2.4 Reducibility to single crossing screening problem

The goal of this paper is to find a sufficient condition for the type set Θ under which given screening problem (Θ, Z, U) can be reduced to a single crossing problem (Θ, X, U) for an allocation subset X of Z without any loss in the principal's maximized payoff.

Definition 2 *For a compact subset $X \subseteq Z$, a screening problem (Θ, Z, U) is* ***reducible to a single crossing problem****, or* ***single-crossing-reducible****, under X if*

1. *The screening problem (Θ, X, U) is a single crossing problem, and*
2. *For every prior belief $\mu \in \Delta(\Theta)$, every optimal IC mechanism under X is also an optimal IC mechanism under Z.*

The second condition says that the optimal mechanisms under Z and X are equivalent in terms of the principal's payoff, and the equivalence is "informationally robust" with respect to distribution over the type set Θ. Hence, when finding an optimal mechanism in the original problem (1), one only needs to solve (2), which is much easier to solve due to the single crossing condition.

A tradeoff occurs when finding a subset X that satisfies both conditions together. The single crossing condition requires an order on the agent's types with which for every pair of elements of X the agent's preference between them satisfies the single crossing property. Hence, a smaller X according to the set inclusion order makes it easier to satisfy the first condition. In contrast, the second condition, the equivalence between optimal IC mechanisms under Z and X, requires X to be rich enough so that the restriction on allocations does not negatively affect the principal in screening the agent.

3 First example

To fix ideas, consider the following example of product line design. A monopolist designs a menu of products characterized by two attributes $(x_1, x_2) \in [0,1]^2$.[4] The cost of producing the good (x_1, x_2) is given as $C(x_1, x_2) = \frac{x_1^2 + x_2^2}{2}$. There are finite N types of buyer whose utility function is symmetric and has a constant elasticity of substitution (CES). That is, each type has a utility function of product defined by

$$u(x) = v \cdot \left(\frac{x_1^r + x_2^r}{2} \right)^{\frac{1}{r}}.$$

for some (v, r) where $v \geq 0$ is the utility from the product $(1,1)$ and $\frac{1}{1-r} \geq 0$ is the elasticity of substitution. Also, the buyer's utility is quasilinear in money. The buyer privately knows his type while the seller only has a prior belief $\mu \in \Delta(\Theta)$ over the types of the buyer. The monopolist seeks to find the optimal selling mechanism that maximizes expected profit.

Generally, the problem is not a single crossing one. This is true even when both values of v and r are increasing according to an order over Θ. To illustrate this, consider the following three type example. In the left panel in Figure 1, the three dots show the buyer's three types with their values of r and v. Here, type θ's value of r is $-\infty$, and thus, the type has Leontief utility function of good. Also, type θ' with $r = 0$ has Cobb Douglas function while type θ''

[4] If the product is a cellphone plan, one can think of x_1 as the amount of data and x_2 as the length of talk that a consumer can use.

with $r = 1$ has a linear utility function with perfect substitutes. The right figure shows their corresponding indifference curves over the set of products where the dots show three possible goods a, b and c.

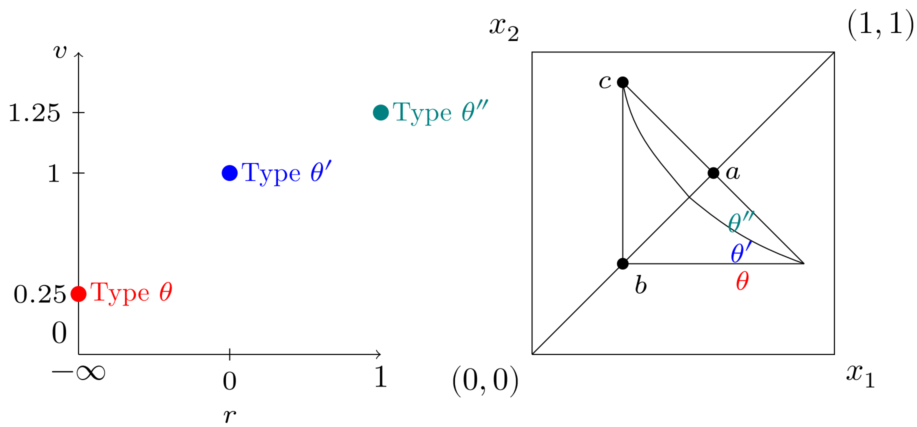

Figure 1

First, consider the two goods a and b. Because both of them have balanced attributes (*i.e.*, $x_1 = x_2$), the size of the payoff difference from the goods is determined solely by the size of v, and thus,

$$u_{\theta''}(a) - u_{\theta''}(b) > u_{\theta'}(a) - u_{\theta'}(b) > u_{\theta}(a) - u_{\theta}(b).$$

Hence, for the single crossing property, the type order $\succeq_\Theta$ has to be chosen as either $\theta'' \succeq_\Theta \theta' \succeq_\Theta \theta$ or $\theta \succeq_\Theta \theta' \succeq_\Theta \theta''$.

However, if one chooses a and c, type θ'' is indifferent between them while good a is more preferred to good c by type θ and θ'. Especially, if v for type θ' is sufficiently larger than v for type θ, so type θ' has a much stronger preference for the good with better attributes among balanced goods, it can be the case that type θ' has a stronger preference for good a over c than type θ. That is,

$$u_{\theta'}(a) - u_{\theta'}(c) > u_{\theta}(a) - u_{\theta}(c) > u_{\theta''}(a) - u_{\theta''}(c).[5]$$

Hence, for the single crossing property, the type order $\succeq_\Theta$ has to be chosen as either $\theta' \succeq_\Theta \theta \succeq_\Theta \theta''$ or $\theta'' \succeq_\Theta \theta \succeq_\Theta \theta'$. Thus, no matter how the goods are ordered, there does not exist a single order of the agent's types that captures the strength of their preference for higher good. This shows that the problem is not a single crossing one.

[5]For example, this holds when $a = (0.6, 0.6)$ and $c = (0.3, 0.9)$.

Despite not having the single crossing property, it turns out that when both v and r are increasing according to a type order, the screening problem is single-crossing-reducible under the set of goods with balanced attributes paired with price, denoted as

$$X := \{(x, x)|x \in [0, 1\} \times \mathbb{R}_+\}.$$

That is, in this case, screening problem with allocation space restricted to X satisfies the single crossing condition and also preserves the seller's maximized profit.

First, I show that the screening problem becomes a single crossing one when the allocation set is restricted to be X. This is because v is the MRS between the balanced increase in the attributes and money at every good with balanced attributes. Hence, if the seller has to sell only goods with balanced attributes, each type's strength of preference for goods with better attributes is determined by only the value of v.[6] Thus, the single crossing property holds when the agent with higher v is higher type and allocation that gives better attributes is higher allocation. Without loss of generality, label each type as n for some $n \in \{1, \cdots, N\}$ so that higher type n has higher v. Because of the single crossing property, it is well-known that if only local IC constraints hold, all the other IC constraints also hold. However, I here use another property of the single crossing problem with the principal's type-independent preference: the solution of the single crossing problem is a solution of a more relaxed problem with fewer constraints only against *downward deviations*. That is, in the relaxed problem, the seller only needs to prevent higher type buyers from reporting a lower type, but this does not strictly increase the seller's payoff.[7]

Next, I show that the seller's maximized profit is preserved even when the allocation set is restricted to be X. Due to the equivalence between the solutions of the single crossing problem and its relaxed one in the previous paragraph, it suffices to prove that this holds in the relaxed problem where the seller only needs to prevent downward deviations.[8] In other words, allocation outside X is not helpful for the principal who only considers downward deviations.

This holds because r is the MRS between larger attribute and the other and the value of r for type n is increasing in n , which means that for the agent with higher n the larger attribute is better substitute for the other. Graphically, this implies that in the space of goods, the indifference curves of all types of the agent are tangent with each other at every point on the

[6]That is, from each good (q, q) for $q \in [0, 1]$, the buyer's utility is given as $u((q, q)) = vq$, which is equivalent to the screening problem addressed by Mussa and Rosen (1978).

[7]The property will be generalized in Section 4.4 while the relaxed problem will be more formally defined in Section 4.2.1.

[8]For more details, see Figure 3 in Section 4.2

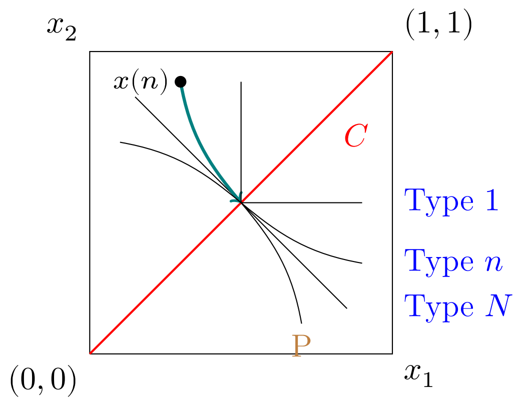

Figure 2

45 degree line. Also, a lower type agent's indifference curve is located above the indifference curve for any higher type. Moreover, by the symmetry of the production cost function, the principal's isocost curve is also tangent with the agent's indifference curves. Figure 2 illustrates the tangency between the curves.

Now, I argue that the tangency condition implies that when only downward deviations need to be prevented, the principal does not have to sell the goods without balanced attributes. Suppose that good $x(n)$ with a price is assigned to type n. Then, if one moves the type n allocation toward 45 degree line with preserving type n agent's payoff and the price, every agent with a higher type will get a lower payoff from the type n allocation, which means that the agent's incentive for any downward deviation is weaker. Moreover, the principal can save on production cost due to its symmetric cost function.

Due to the tangency condition and its implication for the agent's downward deviation incentives, I call the 45 degree line the *incentive contract curve*.[9] The key features and approach here will be generalized in Section 4.

4 Reducibility to single crossing problem

4.1 Statement of the main result

My main result specifies a sufficient condition for the single-crossing-reducibility of (Θ, Z, U) under X. The reduction is possible if with a total order on Θ, X is an incentive contract curve

[9] Contract curve in the Edgeworth box satisfies the similar tangency condition.

and the screening problem (Θ, X, U) is a single crossing problem and has increasing conflict of interest property. Below I define these terms and formally state the main theorem.

First, I define an incentive contract curve in the general setting. In words, an incentive contract curve for a total order on Θ is an allocation subset such that for each allocation z and each type θ, there exists another allocation z' in the subset that satisfies the two conditions: First, for the type θ, z is Pareto dominated by z'; and second, the agent with higher type than θ prefers z to z'.

Definition 3 *An allocation subset $X \subseteq Z$ is an **incentive contract curve** with a total order $\succeq_\Theta$ on Θ if there exists $\pi : Z \times \Theta \to X$ s.t. for all $z \in Z$ and $\theta, \theta' \in \Theta$,*

$$\theta' \succeq_\Theta \theta \implies \begin{cases} U_A(\pi(z,\theta),\theta) \geq U_A(z,\theta) \\ U_P(\pi(z,\theta),\theta) \geq U_P(z,\theta) \\ U_A(\pi(z,\theta),\theta') \leq U_A(z,\theta') \end{cases} \tag{C}$$

Next, for the incentive contract curve X with a total order $\succeq_\Theta$ on Θ, the screening problem (Θ, X, U) has to not only be a single crossing one but also have an increasing conflict of interest with the same order $\succeq_\Theta$. Intuitively, an increasing conflict of interest means that as type goes up the principal is more likely to find it less profitable to give the agent an allocation that is preferable to different types of agent.

Definition 4 *A screening problem (Θ, X, U) has an **increasing conflict of interest** with a total order $\succeq_\Theta$ on Θ if for every $x, x' \in X$ and $\theta, \theta' \in \Theta$ with $\theta' \succeq_\Theta \theta$,*

$$\begin{cases} U_A(x',\theta') \geq U_A(x,\theta') \\ U_A(x',\theta) \geq U_A(x,\theta) \\ U_P(x',\theta') \geq U_P(x,\theta') \end{cases} \implies U_P(x',\theta) \geq U_P(x,\theta). \tag{S}$$

The main result can now be stated as the following.

Theorem 1 *Suppose that there exist a total order $\succeq_\Theta$ on Θ and an allocation subset $X \subseteq Z$ with a total order $\succeq_X$ s.t. with $(\succeq_X, \succeq_\Theta)$*

- *X is an incentive contract curve (Property (C)), and*
- *(Θ, X, U) is a single crossing problem and has increasing conflict of interest (Property (P) and (S)).*

Then, (Θ, Z, U) *is* ***single-crossing-reducible*** *under* X*.*

As will be explained in Remark 2, all of the conditions in the theorem do not conflict with each other. Also, because all these conditions are about the preferences and not affected by the type distribution μ, the single-crossing-reducibility of (Θ, Z, U) is informationally robust with respect to distribution over the type set Θ.

The rest of the section is organized as follows. Section 4.2 describes our approach for proving the main result. In the following subsections, I illustrate the main factors for single-crossing-reducibility. All the formal proofs in this section are relegated to Appendix A.

4.2 Proof outline

4.2.1 Optimal lower IC mechanism

To derive the main result, I first define an *optimal lower IC (LIC) mechanism under* Z with a total order $\succeq_\Theta$ on Θ. Given the total order $\succeq_\Theta$, the mechanism is a solution of the following relaxation of (1), which requires only IC constraints against mimicking lower types according to the order $\succeq_\Theta$, so called *downward IC constraints*:

$$\begin{aligned}
&\max_{h:\Theta\to Z} \mathbb{E}[U_P(h(\theta),\theta)] && \qquad (3)\\
s.t.\ \ & U_A(h(\theta),\theta) \geq U_A(h(\theta'),\theta) && \text{for all } \theta,\theta' \in \Theta \text{ with } \theta \succeq_\Theta \theta'\\
& U_A(h(\theta),\theta) \geq 0 && \text{for all } \theta \in \Theta
\end{aligned}$$

I call an *LIC mechanism* any direct mechanism that satifies all the IR and downward IC constraints.

Similarly, for an allocation subset $X \subseteq Z$, I can define *optimal lower IC (LIC) mechanism under* X with a total order $\succeq_\Theta$ on Θ as the solution of the following problem:

$$\begin{aligned}
&\max_{h:\Theta\to X} \mathbb{E}[U_P(h(\theta),\theta)] && \qquad (4)\\
s.t.\ \ & U_A(h(\theta),\theta) \geq U_A(h(\theta'),\theta) && \text{for all } \theta,\theta' \in \Theta \text{ with } \theta \succeq_\Theta \theta'\\
& U_A(h(\theta),\theta) \geq 0 && \text{for all } \theta \in \Theta
\end{aligned}$$

Again, by using the compactness argument, it can be easily shown that both optimal LIC mechanisms exist.

4.2.2 Payoff equivalence between optimal IC mechanisms

When finding a sufficient condition for the single-crossing-reducibility of (Θ, Z, U) under X, the main challenge is finding when the second condition in Definition 2, the payoff equivalence between the optimal IC mechanisms under Z and X, holds. To do so, I use the principal's ranking of the four optimal mechanisms: optimal LIC and IC mechanisms under Z and X with a total order $\succeq_\Theta$ on the type space Θ.

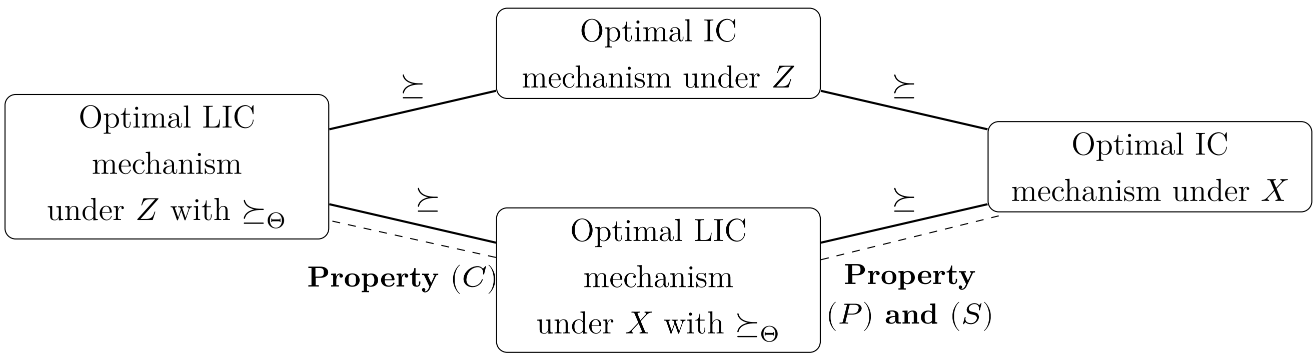

Figure 3

In Figure 3, the four preference relations show the ranking of the mechanisms based on the principal's expected payoffs. First, given the allocation set Z or X and a total order $\succeq_\Theta$ on Θ, the principal always gets a weakly larger expected payoff from optimal LIC mechanism than optimal IC mechanism. This is because the former, which requires only downward IC constraints, is a relaxation of the latter. Next, given the set of IC constraints that contains all the ICs or only downward ICs, optimal mechanism under Z always gives a weakly larger payoff to the principal than optimal mechanism under X. This is again because the latter under the allocation subset Z is a solution of a more relaxed one than the former under $X \subseteq Z$. More specifically, the restriction on the feasible allocations can negatively affect the principal in screening the agent.[10]

The preference relations in Figure 3 imply that if optimal LIC mechanism under Z with $\succeq_\Theta$ is equivalent to optimal IC mechanism under X, all four optimal mechanisms are equivalent. Hence, for the payoff equivalence between optimal IC mechanisms under X, it suffices to find a condition under which the following equivalences hold together:

[10]For example, in two good monopoly when the values are additive and perfectly negatively correlated, full surplus extraction is possible by bundling the two goods. However, if under allocation subset X bundling is not allowed, the seller, who has to sell the goods separately, cannot help paying some information rent, which is less profitable for her.

1. An optimal LIC mechanism under X is an optimal LIC mechanism under Z.

2. An optimal IC mechanism under X is an optimal LIC mechanism under X.

Remark 1 *If the four optimal mechanisms are equivalent, this also implies that the four sets of optimal mechanisms have the following set inclusion relations as illustrated by Figure 4. Especially, the set of all optimal IC mechanisms under X is a subset of the set of all optimal IC mechanisms under Z.*

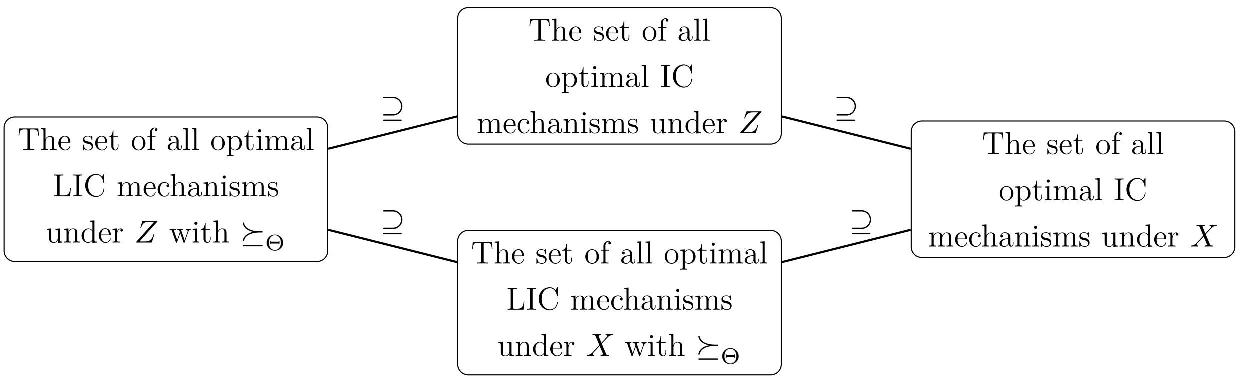

Figure 4

In Section 4.3, I show that incentive contract curve X (Property (C)) guarantees the first equivalence, the equivalence between optimal LIC mechanisms under X and Z. In Section 4.4, I show that single crossing problem (Θ, X, U) with increasing conflict of interest (Property (P) with (S)) guarantees the second equivalence, the equivalence between optimal IC and LIC mechanism under X and Z. Then, because all the conditions are independent of the type distribution μ, the equivalence between optimal IC mechanisms under X and Z is informationally robust with respect to distribution over the type set Θ.

4.3 Equivalence between optimal LIC mechanisms under Z and X

In this subsection, I show that optimal LIC mechanisms under Z and X are equivalent if X is an incentive contract curve (Property (C)). First, recall Property (C) in Definition 3:

There exists $\pi : Z \times \Theta \to X$ s.t. for all $z \in Z$ and $\theta, \theta' \in \Theta$,

$$\theta' \succeq_\Theta \theta \implies \begin{cases} U_A(\pi(z,\theta),\theta) \geq U_A(z,\theta) \\ U_P(\pi(z,\theta),\theta) \geq U_P(z,\theta) \\ U_A(\pi(z,\theta),\theta') \leq U_A(z,\theta') \end{cases} \tag{C}$$

Intuitively, the property says that allocations outside the incentive contract curve are not useful for a principal who only needs to prevent the agent's downward deviations. That is, if an LIC mechanism assigns allocation z to type θ, by replacing z with $\pi(z,\theta)$ the principal's payoff can be increased without violating any downward IC constraint. Therefore, for an incentive contract curve X, optimal LIC mechanisms under Z and X are equivalent in terms of the principal's payoff.

Proposition 1 *Suppose that X is an incentive contract curve with a total order $\succeq_\Theta$ on Θ. Then, for all type distributions $\mu \in \Delta\Theta$, with $\succeq_\Theta$*

- *Optimal LIC mechanisms under Z and X give the principal the same expected payoff.*
- *The set of all optimal LIC mechanisms under X is a subset of the set of all optimal LIC mechanisms under Z.*

It is worth noting that for every total order on the type set, an incentive contract curve always exists. This is because the allocation set Z itself is an incentive contract curve for π defined by $\pi(z,\theta) = z$ for every $z \in Z = X$ and $\theta \in \Theta$. However, for the single crossing condition of (Θ, X, U), one needs to find a sufficiently small incentive contract curve according to the set inclusion order. Appendix B provides one way to construct such an incentive contract curve to implement the main result.

In the example of Section 3, it can be shown that when type n's value of r is nondecreasing in n, $X = \{(q,q) | q \in [0,1]\} \times \mathbb{R}_+$ is an incentive contract curve with the order $\geq$ on $\Theta = \{1, 2, \cdots, N\}$.[11] Defining $\pi : Z \times \Theta \to X$ s.t. for each n and each allocation $((q_1,q_2),p) \in Z = [0,1]^2 \times \mathbb{R}_+$,

$$\pi(((q_1,q_2),p),n) = ((q_0,q_0),p)$$

where (q_0,q_0) is the good with balanced attributes that gives the same payoff to type θ as (q_1,q_2), *i.e.*, $q_0 = u_n(q_1,q_2)$. Hence, the first inequality about payoffs in Property (C) holds.

[11]Recall that each type is labeled as one of the numbers from 1 to N so that type n's value of v is increasing in n.

As illustrated by Figure 2, the other two inequalities hold because of the tangency condition between the indifference curves of all players in the product space.

4.4 Equivalence between optimal LIC and IC mechanisms under X

In this section, I show that in a single crossing screening problem, optimal LIC and IC mechanisms are equivalent if the problem satisfies increasing conflict of interest (Property (S)). In other words, any optimal IC mechanism is a solution of a more relaxed problem that only requires the IR and downward IC constraints. The property of increasing conflict of interest is about how the relation between principal and agent's preferences change as the type increases. First, recall Property (S) in Definition 4:

For every $x, x' \in X$ and $\theta, \theta' \in \Theta$ with $\theta' \succeq_\Theta \theta$,

$$\begin{cases} U_A(x',\theta') \geq U_A(x,\theta') \\ U_A(x',\theta) \geq U_A(x,\theta) \\ U_P(x',\theta') \geq U_P(x,\theta') \end{cases} \implies U_P(x',\theta) \geq U_P(x,\theta). \tag{S}$$

The property says that for any pair of types and any pair of elements of X, if both types of agent have the same preference over the allocations, and the principal with the higher type agent also shares the same preference over the allocations, then does the principal with the lower type. Roughly, for a pair of allocations that cannot be used to screen the types of agent, the principal with the lower type is more likely to have the same preference over the pair of allocations with those types of agent, which implies less conflict of interest between the principal and the agent at the lower type. Hence, the principal does not have an incentive to violate the upward IC constraint by assigning the more preferable allocation only to the higher type even when only downward IC constraints need to be satisfied.

For an intuition for the equivalence result, consider the case of binary type with binary allocation. Suppose that the property does not hold for the two allocations x and x'. That is, the principal with the low type strictly prefers x to x' while the others, both types and the principal with the high type, prefers x' to x. Then, if only downward IC mechanism needs to be satisfied, the principal optimally assigns x' to the high type and x to the low type rather than giving both types the same allocation, either x or x'. However, this does not satisfy the upward IC constraint. Because of the single crossing property of the agent's preference, this logic of binary type example can be easily extended to the general type case. The formal proof of the proposition is relegated to Appendix A.2.

Proposition 2 *Suppose that (Θ, X, U) is a single crossing screening problem and has increasing conflict of interest with $(\succeq_X, \succeq_\Theta)$. Then, for all type distributions $\mu \in \Delta\Theta$, every optimal IC mechanism under X is an optimal LIC mechanism under X with $\succeq_\Theta$.*

The property of increasing conflict of interest always holds if the principal's preference is type-independent. In the case of type dependent preference of the principal, the property also holds if the players' preferences U, which integrates the preferences of the principal and the agent U_P and U_A over X, has the single crossing property with an order of the players satisfying some conditions, which is addressed in Appendix C. Especially, Appendix D shows that when the players are ordered in a particular fashion, an optimal IC mechanism under X assigns the same allocation to every type of agent. This leads to the optimality of no screening in the original problem if X is an incentive contract curve. Also, in Appendix G, I show the property can be further weakened under some assumptions, such as the existence of money transfers.

Remark 2 *The three conditions in the main result, Property (C) for X being an incentive contract curve, Property (P) for the single crossing property and Property (S) for increasing conflict of interest of (Θ, X, U), do not conflict with each other. (C) is about the players' preferences for elements over non-elements of X while (P) and (S) are about their' preferences over X. Moreover, (P) is only about how type affects the agent's preference while (S) is about how type affects the relations between the principal and the agent's preferences.*

5 Optimality of nested mechanism

In this section, I consider a case that is particularly easy to apply the main result. In this case, the principal and the agent have linear payoff functions, and a dominance relation, similar to a stochastic dominance, is defined over allocations. These conditions facilitate finding sufficient conditions for single-crossing-reducibility. First, the linearity of the payoff functions allows us to guess that an incentive contract curve X can be located on the boundary of the allocation set Z. Moreover, combined with the dominance relation over allocations, it enables us to find simple sufficient conditions for Property (C) with X having a desirable ordering property. By using the ordering property of X, I find conditions for Property (P) and (S), and this also leads to a *nested structure* of optimal IC mechanism. That is, all the available allocations in the optimal mechanism are totally ordered by the standard componentwise order. Although this case may seem stylized, it will turn out that many well-known screening problems can be reduced to problems that satisfy the conditions. In the next section, I will apply the results here to various economic settings such as multiple good monopoly, persuasion and delegation. All the formal proofs are relegated to Appendix E.

5.1 Additional setup

Each allocation $z := (z^+, z^-)$ consists of values for positive components $z^+ : C^+ = \{1, \cdots, M\} \to \mathbb{R}_+$ and those for negative components $z^- : C^- = \{1, \cdots, N\} \to \mathbb{R}_+$. For each positive component m, every type of the agent prefers larger $z^+(m)$ while for each negative component n, each type prefers smaller $z^-(n)$. Allocation $z \equiv 0$ is type-independent outside option.

Next, I define the following orders on allocations. First, for $z \in Z$, define $s_z^+(m) := \sum_{m' \leq m} z^+(m'),\ s_z^-(n) := \sum_{n' \leq n} z^-(n')$.

Definition 5 *For $k \in \{0, 1, 2\}$, the kth* ***dominance order*** *$\succeq_k$ is a partial order on Z s.t.*

- $(\mathbf{k = 0})$ $z' \succeq_0 z \Leftrightarrow$ *For all* $c \in C,\ z'(c) \geq z(c)$
- $(\mathbf{k = 1})$ $z' \succeq_1 z \Leftrightarrow$ *For all* $m,\ n,\ s_{z'}^+(m) \geq s_z^+(m),\ s_{z'}^-(n) \geq s_z^-(n)$.
- $(\mathbf{k = 2})$ $z' \succeq_2 z \Leftrightarrow$ *For all* $m,\ n,$

$$\sum_{m' \leq m} s_{z'}^+(m') \geq \sum_{m' \leq m} s_z^+(m'),\ \sum_{n' \leq n} s_{z'}^-(n') \geq \sum_{n' \leq n} s_z^-(n')$$

The 0th dominance order is the standard component-wise order. The first and second dominance orders are similar to the first and second stochastic dominance orders. Especially, for probability distributions z^+ and z'^+ over C^+ and z^- and z'^- over C^-, that z' first (second) order dominates by z means that z^+ and z^- respectively first (second) order stochastically dominate by z'^+ and z'^-.

By using the dominance orders, I define the allocation set. Let Z be a lower contour set for a fixed allocation $\bar{z} := (\bar{z}^+, \bar{z}^-)$ and an order $\succeq_k$ for a fixed $k \in \{0, 1, 2\}$:

$$Z := \{z = (z^+, z^-) \in \mathbb{R}_+^{C^+} \times \mathbb{R}_+^{C^-} | \bar{z} \succeq_k z\}$$

I denote $\bar{z}_0^+ := \max \operatorname{supp} \bar{z}^+$ and $\bar{z}_0^- := \max \operatorname{supp} \bar{z}^-$ which are respectively the maximum positive and negative components that $\bar{z}$ have nonzero values.

The dominance constraint arises for different reasons in economic problems: For example, in multiple good monopoly the 0th dominance order stems from a capacity constraint related to a unit supply for each good while the first dominance order stems from a feasibility condition for the probabilistic bundling. The second dominance order is derived from the incentive compatibility constraint in a delegation problem and the constraint for the distribution of posteriors in a persuasion problem.

Lastly, I assume that the principal and the agent's payoffs are linear in allocation. Denote as $u_P^+(m,\theta)$ and $u_A^+(m,\theta)$ the principal and the agent's marginal benefits of the positive component m at θ and as $u_P^-(n,\theta)$ and $u_A^-(n,\theta)$ their marginal losses of the negative component n at θ. According to the definition of positive and negative components, $u_A^+(m,\theta)$ and $u_A^-(n,\theta)$ are nonnegative for every m, n and θ. Additionally, I assume that $u_A^-(n,\theta)$ is strictly positive for every n and θ. Then, the principal and the agent's payoff functions can be defined as

$$U_P(z,\theta) = \sum_{m=1}^{M} z^+(m)u_P^+(m,\theta) - \sum_{n=1}^{N} z^-(n)u_P^-(n,\theta) = U_P^+(z,\theta) - U_P^-(z,\theta)$$
$$U_A(z,\theta) = \sum_{m=1}^{M} z^+(m)u_A^+(m,\theta) - \sum_{n=1}^{N} z^-(n)u_A^-(n,\theta) = U_A^+(z,\theta) - U_A^-(z,\theta)$$

where $U_P^+(z,\theta)$, $U_P^-(z,\theta)$, $U_A^+(z,\theta)$ and $U_A^-(z,\theta)$ are the principal and the agent's total benefits and losses from the positive and negative components, respectively.

5.2 Reduction to single crossing screening problem

5.2.1 Upper boundary set of Z

It is well-known that the contract curve in the Edgeworth box for an economy with two linear utility functions is either the right and bottom edges or the left and top edges of the box unless the utility functions are same. Likewise, due to the linearity of U, one can guess that the smallest incentive contract curve can be on the boundary of Z. Moreover, the following allocation subset, which is analogous to the two fold product of the contract curve, will be the incentive contract curve X in my analysis.

Definition 6 *For $Z = \{z = (z^+, z^-) \in \mathbb{R}_+^{C^+} \times \mathbb{R}_+^{C^-} | \bar{z} \succeq_k z\}$, upper boundary set of Z, $\hat{bd}(Z) \subseteq Z$, is the set of all allocations such that for some $1 \leq \bar{m} \leq M$ and $1 \leq \bar{n} \leq N$*

- *$z^+(m) = \bar{z}^+(m)$ and $z^-(n) = \bar{z}^-(n)$ for $m < \bar{m}$ and $n < \bar{n}$*
- *$z^+(m) = 0 = z^-(n)$ for $m > \bar{m}$ and $n > \bar{n}$.*

The definition says that each of the positive and negative components of elements of $\widehat{bd}(Z)$ takes the following form: whenever the element has a strictly positive value for component k, it has the same values for all the lower components than k as the upper bound $\bar{z}$. This implies that the positive and negative components of allocations in $\widehat{bd}(Z)$ are respectively totally ordered by the standard component wise order $\geq$. More specifically, for every $z, z' \in \widehat{bd}(Z)$, either $z^+ \geq z'^+$ or $z'^+ \geq z^+$. In other words, either $z^+(m) \geq z'^+(m)$ for every m or $z'^+(m) \geq z^+(m)$ for every m. Likewise, for the negative components either $z^- \geq z'^-$ or $z'^- \geq z^-$.

Hence, the following order $\succeq_{bd}$ on $\widehat{bd}(Z)$ is a total order:

$$z' \succeq_{bd} z \Leftrightarrow z'^+ \geq z^+ \tag{5}$$

In the following subsections, I find the sufficient condition for single-crossing-reducibility of (Θ, Z, U) under $X = \widehat{bd}(Z)$. According to Theorem 1, I need to find the conditions for the following with a total order $\succeq_\Theta$ on Θ:

1. Single crossing problem (Θ, X, U) with $(\succeq_{bd}, \succeq_\Theta)$ (Property (P))

2. Increasing conflict of interest of (Θ, X, U) with $\succeq_\Theta$ (Property (S))

3. Incentive contract curve X with $\succeq_\Theta$ (Property (C))

5.2.2 Single crossing condition

First, I establish the single crossing condition on the upper boundary set $\widehat{bd}(Z)$. The following condition, *increasing marginal rates of substitution*, means that higher type agent has a higher marginal benefit from a positive component relative to the marginal loss from a negative one.

Definition 7 *A type set Θ has* ***increasing marginal rates of substitution (MRS)*** *with a total order $\succeq_\Theta$ on Θ if one of the followings holds:*

1. $C^- \neq \emptyset$; and for every m and n, $\frac{u_A^+(m,\theta)}{u_A^-(n,\theta)}$ is nondecreasing in type.

2. $C^- = \emptyset$; and for every m and $\theta' \succeq_\Theta \theta$, $u_A^+(m, \theta') > 0$ whenever $u_A^+(m, \theta) > 0$.

Increasing MRS condition guarantees the single crossing condition of $(\Theta, \widehat{bd}(Z), U)$. This is due to the totally ordered positive and negative components of allocations in the upper boundary set $\widehat{bd}(Z)$. Moreover, under this condition optimal IC mechanism under the upper boundary set has a nested form, that is, all the available allocations in the optimal mechanism are totally ordered by the standard componentwise order $\geq$ over $\mathbb{R}_+^{M+N} \supseteq Z$.

Proposition 3 *Suppose that a type set Θ has increasing MRS with $\succeq_\Theta$. Then, for $X = \widehat{bd}(Z)$, (Θ, X, U) is a single crossing one with $(\succeq_{bd}, \succeq_\Theta)$. Moreover, the optimal IC mechanism under X has a nested form.*

5.2.3 Sufficient condition for the increasing conflict of interest

In this section, I provide a set of sufficient conditions for increasing conflict of interest of the screening problem $(\Theta, \widehat{bd}(Z), U)$. First, the property holds if the principal's payoff is independent of type, that is, for every m and n, $u_P^+(m, \theta)$ and $u_P^-(n, \theta)$ are constant for θ. For example, this holds in multiple good monopoly problems where the seller maximizes the revenue.[12]

When the principal's payoff is type dependent, a set of sufficient conditions for the increasing conflict of interest property can be easily derived by using the linearity of payoff functions. Here are some examples. First, when the agent has a positive marginal payoff from every component, the increasing conflict of interest holds if the principal's marginal benefit is nonincreasing in type.[13] If the principal has a positive marginal payoff from every component, the property holds if the principal's preferences have the single crossing property.[14]

Proposition 4 *Suppose that type set Θ has increasing MRS with $\succeq_\Theta$. Then, for $X = \widehat{bd}(Z)$, (Θ, X, U) has the increasing conflict of interest if one of the following conditions holds:*

1. *$C^- = \emptyset$; and for every m, $u_p^+(m, \theta)$ is nonincreasing in type; and $u_p^+(m, \theta) \leq 0$ whenever $u_A^+(m, \theta) = 0$.*

2. *For every m, n, and θ, $u_P^+(m, \theta) > 0 > u_P^-(n, \theta)$; and for every m and n $\frac{u_P^+(m,\theta)}{u_P^-(n,\theta)}$ is nondecreasing in type.*

5.2.4 Sufficient conditions for incentive contract curve $\widehat{bd}(Z)$

So far, I have found conditions for the single crossing condition and the increasing conflict of interest of $(\Theta, \widehat{bd}(Z), U)$ with a pair of total orders $(\succeq_\Theta, \succeq_{bd})$. In this section, I provide a condition that makes sure that the upper boundary set $\widehat{bd}(Z)$ is an incentive contract curve

[12]See Section 6.1 for this case.

[13]See Section 6.3 for this case.

[14]Appendix G provides a weaker version of the increasing conflict of interest, and it shows that optimal IC and LIC mechanisms are equivalent in a single crossing problem whenever the principal's marginal payoff from every component is positive. See Proposition 7 in Section 6.2 for a concrete example.

with $\succeq_\Theta$. I first choose the mapping π_{bd} from Z to $X = \hat{bd}(Z)$ by

$$\pi_{bd}(z,\theta) \in \{x \in X | (U_A^+(x,\theta), U_A^-(x,\theta)) = (U_A^+(z,\theta), U_A^-(z,\theta))\}.$$

Because the mapping π_{bd} has to preserve the total benefit and loss from the positive components and the negative components, it might not be well-defined. The sufficient condition for its well-definedness can be obtained as the following due to the same logic of a well-known fact about stochastic dominance.[15] Here, to strengthen the result in the case when $\bar{z}$ does not have full support, I define that a real-valued function u over $Y \subset \mathbb{N}$ is *quasi-decreasing* with respect to cutoff $\bar{y} \in Y$ when u is nonincreasing for $y \leq \bar{y}$ and

$$y' \leq \bar{y} \leq y \implies u(y') \geq u(y).$$

Lemma 1 *Suppose that $X = \hat{bd}(Z)$. Then, $\pi_{bd} : Z \times \Theta \to X$ is well-defined for each of the following three cases,*

- *Z with the **0th** dominance order*
- *Z with the **1st** dominance order and*

  $$\text{for every } \theta,\ u_A^+(m,\theta) \text{ and } u_A^-(n,\theta) \text{ are quasi-decreasing in } m \text{ and } n \text{ with respect to } \bar{z}_0^+ \text{ and } \bar{z}_0^- \tag{M_1}$$

- *Z with the **2nd** dominance order and*

  $$\text{for every } \theta,\ u_A^+(m,\theta) \text{ and } u_A^-(n,\theta) \text{ are quasi-decreasing in } m \text{ and } n \text{ with respect to } \bar{z}_0^+ \text{ and } \bar{z}_0^- \text{ and convex in } m \text{ and } n \text{ for } m \leq \bar{z}_0^+ \text{ and } n \leq \bar{z}_0^-. \tag{M_2}$$

Next, for the upper boundary set being an incentive contract curve, I find a sufficient condition for Property (C) when $\pi = \pi_{bd}$. To state the condition, the following ratios between marginal payoffs need to be defined.

Definition 8 *For type θ and θ', the ratio between marginal benefits and the ratio between marginal losses of type θ and θ' agents are defined as the following:*

$$r^+_{\theta,\theta'}(m) := \frac{u_A^+(m,\theta)}{u_A^+(m,\theta')},\ r^-_{\theta,\theta'}(n) := \frac{u_A^-(n,\theta)}{u_A^-(n,\theta')}$$

[15] For cumulative distribution functions F and G over a set Y, F first (second) order stochastically dominates G iff $\mathbb{E}_F[u(y)] \geq \mathbb{E}_G[u(y)]$ for all nondecreasing (nondecreasing and concave) functions u.

Similarly, for type θ, those ratios between the principal and the agent with type θ are defined as

$$r^+_{P,\theta}(m) := \frac{u^+_P(m,\theta)}{u^+_A(m,\theta)},\ r^-_{P,\theta}(n) := \frac{u^-_P(n,\theta)}{u^-_A(n,\theta)}$$

Property (C) holds if for every type θ and allocation z there exists a path from z to $\pi_{bd}(z,\theta)$ along which type θ's payoff is preserved, decreases all higher types' payoffs, and increases the payoff of the principal with type θ. It can be shown that such a path exists if the marginal benefit (loss) of type θ is more (less) strongly decreasing and convex in component than that of higher type and less (more) than that of the principal with type θ in the following sense.

Proposition 5 *$X = \hat{bd}(Z)$ is an incentive contract curve with a total order $\succeq_\Theta$ on Θ if one of the following conditions hold:*[16]

- *Z is defined with the 0th dominance order; and*

  for every $\theta' \succeq_\Theta \theta$, $r^+_{\theta,\theta'}(m)$, $-r^-_{\theta,\theta'}(n)$, $r^+_{P,\theta}(m)$ and $-r^-_{P,\theta}(n)$ are quasi-decreasing in m and n with respect to $\bar{z}^+_0$ and $\bar{z}^-_0$. $\quad(R_1)$

- *Z is defined with the 1st dominance order; and (M_1) and (R_1) hold*

- *Z is defined with the 2nd dominance order; and (M_2) holds; and*

  for every $\theta' \succeq_\Theta \theta$, $r^+_{\theta,\theta'}(m)$, $-r^-_{\theta,\theta'}(n)$, $r^+_{P,\theta}(m)$ and $-r^-_{P,\theta}(n)$ are quasi-decreasing in m and n with respect to $\bar{z}^+_0$ and $\bar{z}^-_0$ and convex in m and n for $m \le \bar{z}^+_0$ and $n \le \bar{z}^-_0$.[17] $\quad(R_2)$

[16] In fact, all of these conditions only need to hold for the pair of adjacent types θ and θ' because it suffices to show that $\pi_{bd}(z,\theta)^+$ and $\pi_{bd}(z,\theta)^-$ are nondecreasing in θ, which only needs to be checked locally.

[17] When $\bar{z}$ has the full support, (R_2) can be weakened as the following: for every $\theta' \succeq_\Theta \theta$,

$$\frac{u^+_A(m,\theta)}{u^+_A(m,\theta')}, \frac{\Delta u^+_A(m,\theta)}{\Delta u^+_A(m,\theta')}, \frac{u^+_P(m,\theta)}{u^+_A(m,\theta)}, \frac{\Delta u^+_P(m,\theta)}{\Delta u^+_A(m,\theta)} \text{ are nonincreasing in } m \text{, and} \qquad (\hat{R}_2)$$
$$\frac{u^-_A(n,\theta')}{u^-_A(n,\theta)}, \frac{\Delta u^-_A(n,\theta')}{\Delta u^-_A(n,\theta)}, \text{-}\frac{u^-_P(n,\theta)}{u^-_A(n,\theta)}, \text{-}\frac{\Delta u^-_P(n,\theta)}{\Delta u^-_A(n,\theta)} \text{ are nonincreasing in } n$$

where $\Delta u^+_A(m,\theta) = u^+_A(m,\theta) - u^+_A(m+1,\theta)$, $\Delta u^-_A(n,\theta) = u^-_A(n,\theta) - u^-_A(n+1,\theta)$, $\Delta u^+_P(m,\theta) = u^+_P(m,\theta) - u^+_P(m+1,\theta)$, and $\Delta u^-_P(n,\theta) = u^-_P(n,\theta) - u^-_P(n+1,\theta)$.

5.2.5 Sufficient conditions for single-crossing-reducibility

Combining all the conditions for the single-crossing-reducibility, we obtain the following result. Due to Proposition 3, this also implies that there is an optimal mechanism under Z that has a nested form.

Theorem 2 *For $X = \widehat{bd}(Z)$, suppose that with a total order $\succeq_\Theta$ the type set Θ has increasing MRS and increasing conflict of interest, and one of the following conditions also holds:*

- *Z is defined with the 0th dominance order; and (R_1) holds*
- *Z is defined with the 1st dominance order; and (M_1) and (R_1) hold*
- *Z is defined with the 2nd dominance order; and (M_2) and (R_2) hold.*

Then, (Θ, Z, U) is single-crossing-reducible under X and there exists an optimal IC mechanism under Z that has a nested form.

All the results that I have derived in the section also hold even when C^+ and C^- are compact subsets of $\mathbb{R}$. Instead of the summations, Riemann integrals can be used to define dominance orders, and then, all the results including Theorem 2 with increasing and convex properties directly extend to the continuous case.

6 Applications

Theorem 2 in the previous section can be used to derive new sufficient conditions for the optimality of simple mechanisms in various settings: upgrade pricing in a multiple good monopoly problem and no screening with same cap in a delegation problem. Also, I can easily reproduce optimality conditions in the existing literature: pure bundling in the multiple good monopoly (Haghpanah and Hartline, 2021) and nested interval structure of acceptance sets in the persuasion problem (Guo and Shmaya, 2019).

6.1 Multiple good monopoly

Now, I first apply our result to a standard setting of multiple good monopoly. By using the equivalence result, I recover the optimality result of *pure bundling* that offers only the grand bundle in Haghpanah and Hartline (2021). Then, I extend the result into *upgrade pricing* where all the available bundles are ordered by the set inclusion.[18]

Setup

There are two players: a buyer and a seller. The seller sells d products with zero production cost. The grand bundle $b^* = \{1, 2, \cdots, d\}$ contains all the products, and the set of all bundles is $B = \{b | b \subseteq b^*\}$. I allow the seller to choose any randomized bundle to sell from ΔB. Hence, the allocation set Z is equal to the set of all the pairs of a randomized bundle and its relative price to the maximum price $\bar{p}$, that is,

$$Z = \Delta(B) \times [0, 1].^{19}$$

If the buyer with type $\theta \in \Theta$ purchases a bundle $b \in B$ with price $\alpha\bar{p}$ for some $\alpha \in [0, 1]$, his payoff can be written as the following by using our notation:

$$U_A((b, \alpha), \theta) = v_\theta(b) - \alpha\bar{p}.$$

where $v_\theta : B \to \mathbb{R}_+$ is the value function of the buyer with type θ. The payoff function can be extended to every randomized bundle $a \in \Delta B$ by taking expectation over goods:

$$U_A((a, \alpha), \theta) = \mathbb{E}_a[v_\theta(b)] - \alpha\bar{p}.$$

The buyer's payoff from the outside option $(a, \alpha) = (0, 0)$ or not buying anything is normalized to zero. The seller's payoff is equal to the price if the buyer accepts the offer and zero otherwise:

$$U_P((a, \alpha), \theta) = \alpha\bar{p}.$$

Pure bundling

First, I reproduce the sufficient condition for pure bundling in Haghpanah and Hartline (2021).

To do so, I redefine the allocation set Z as a lower contour set of $\bar{z} := (\mathbb{1}_{b^*}, 1)$ with respect to the first dominance order. That is, for $C^- = \{1\}$ and $C^+ = B$ where $M := |B|$ and b^* is its first component,

$$Z = \Delta(B) \times [0, 1] = \{z = (z^+, z^-) \in \mathbb{R}^{C^+} \times \mathbb{R}^{C^-} | \bar{z} := (e_1, \bar{p}) \succeq_1 z\}$$

[18] I use the definition of upgrade pricing in Bergemann, Bonatti, Haupt, and Smolin (2021).

[19] Here, it is without loss of generality to set the maximum price to be the maximum among the valuations that each type of agent can have for each bundle.

Here, e_1 is a unit vector in $[0,1]^M$ with the first component equal to 1. All the bundles other than the grand bundle can be labeled from 2 to M arbitrarily. Note that the first dominance relation comes from the feasibility condition for probabilistic bundling.

For the allocation set Z, all the player's payoffs are linear in allocations. More precisely, type θ agent's marginal benefit from each m or a bundle labeled by m is his valuation for the bundle while the marginal loss can be set by the maximum price $\bar{p}$:

$$u_A^+(m,\theta) = v_\theta(m),\ u_A^-(1,\theta) = \bar{p}.$$

And the principal's payoff is equal to the revenue:

$$u_P^+(m,\theta) = 0,\ u_P^-(1,\theta) = -\bar{p}.$$

According to the definition of Z, the upper boundary set $\widehat{bd}(Z)$ is the set of allocations that only assigns the buyer the grand bundle b^*:

$$X := \widehat{bd}(Z) = \{z \in Z | z^+(m) = 0 \text{ for every } 1 < m \leq M\}$$

Therefore, if the screening problem (Θ, Z, U) is single-crossing-reducible under X, a solution for the problem is the optimal selling mechanism for (Θ, X, U) that is equivalent to the single good monopoly problem where the only good is the grand bundle. Because it is well known that posting a single price is optimal in the single good monopoly, this implies that pure bundling is optimal for (Θ, Z, U). Hence, in order to find when pure bundling is optimal, it suffices to find a sufficient condition for the single-crossing-reducibility of (Θ, Z, U) under X, which is derived by Theorem 2.

Corollary 1 ***(Haghpanah and Hartline, 2021)*** *Pure bundling is optimal for all distribution $\mu \in \Delta\Theta$ if for every $b \in B$, $v_\theta(b^*) \geq v_\theta(b)$ and there exists a total order on Θ such that for every θ, θ' with $\theta' \succeq_\Theta \theta$*

$$v_{\theta'}(b^*) \geq v_\theta(b^*) \tag{6}$$

$$\frac{v_{\theta'}(b^*)}{v_\theta(b^*)} \leq \frac{v_{\theta'}(b)}{v_\theta(b)} \tag{7}$$

According to Theorem 2, (Θ, Z, U) is single-crossing-reducible under X if the four conditions, increasing MRS, increasing conflict of interest, (M_1), and (R_1), hold. Increasing MRS is equivalent to (6), monotone MRS between the grand bundle value and money while the increasing conflict of interest holds because of the seller's type-independent payoff. (M_1) is equivalent to that $v_\theta(b^*) \geq v_\theta(b)$ for every $b \in B$, so called the *free disposal property*, because $u_A^+(m,\theta) = v_\theta(m)$. Lastly, (R_1) is equivalent to (7), monotone marginal rates of substitution between the grand bundle value and all smaller bundles.[20]

The set of conditions for the optimality of pure bundling is equivalent to the condition found by Haghpanah and Hartline (2021). As in their papers, it can be easily shown that the condition is necessary and sufficient condition for the robust optimality of pure bundling with respect to distribution over the type set. Also, the condition can be readily extended to their more general sufficient condition, the positive correlation between the grand bundle value and relative values for smaller bundles, by using the decomposition method of Strassen (1965).

The main difference from the derivation in Haghpanah and Hartline (2021) is that our approach uses a primal approach rather than their dual approach.[21] In their paper, the optimality result is derived by using duality to construct the Lagrangian variables that are non-zero only for the downward IC constraints and make the ironed virtual valuation nondecreasing in grand bundle value. However, this construction method cannot be directly applied to find the condition for the optimality of the menu containing multiple items. In contrast, my result allows to not only reproduce the condition for the optimality of pure bundling more easily but also prove optimality for more general form of selling mechanisms such as upgrade pricing, which will be explained below. Due to its more general form, the optimality condition holds for some additive valuation functions of the buyer, unlike the condition for pure bundling.

The ease of deriving the optimality condition compared to dual approach is mainly because our result enables us to separate the process of finding a solution from the original screening problem without single crossing property into two processes: one for showing the equivalence between the original problem and a single crossing screening problem(here, equivalent to single good monopoly) and the other for finding the solution from the reduced single crossing problem. This separation of the processes makes the result not only more applicable to various settings, but also makes it easier to find the solution by reducing the screening problem to a single crossing problem.

Upgrade pricing

[20]More specifically, $r^+_{\theta,\theta'}(m) = \frac{u_A^+(m,\theta)}{u_A^+(m,\theta')} = \frac{v_\theta(m)}{v_{\theta'}(m)}$ is the ratio between type θ and θ''s valuations for the mth bundle when the first bundle is the grand bundle. Hence, $r^+_{\theta,\theta'}(m)$ is quasi-decreasing in m with respect to $\bar{z}_0^+ = 1$ if (7) holds.

[21]Yang (2021) also uses equivalent primal approach to extends this result to the case when both bundling and quality discrimination are allowed.

Next, I restrict attention to the case when the value function v_θ is additive.[22] Then, I apply my result to find a sufficient condition about the type set for the optimality of upgrade pricing, which is independent of distribution over the type set.

For an additive value function v_θ, the value of buyer with type $\theta \in \Theta$ for each randomized bundle $a \in \Delta B$ is determined by his value for each good l, $v_\theta(l)$, and the total expected probability with which the buyer will get good i: $v_\theta(a) = \sum_{l=1}^{d} v_\theta(l)\mathbb{E}_a[\mathbb{1}_{l\in b}]$. Therefore, without loss of generality the set of all possible bundle ΔB can be replaced with $[0,1]^d$ whose each element $(q_1, q_2, \cdots, q_d)$ means that it gives the buyer each good l with probability q_l. Thus, the allocation set Z is equal to $[0,1]^d \times [0,1]$, which is the lower contour set of the constant function $\bar{z} := \mathbb{1}$ with respect to the 0th dominance order: For $C^+ = \{1, \cdots, d\}$ and $C^- = \{1\}$,

$$Z = [0,1]^d \times [0,1] = \{z = (z^+, z^-) \in \mathbb{R}^{C^+} \times \mathbb{R}^{C^-} | \bar{z} := \mathbb{1} \succeq_0 z\}$$

Again, for Z all the players' payoffs are linear in allocations with the following marginal benefit and loss from each component of allocation:

$$u_A^+(m,\theta) = v_\theta(m),\ u_A^-(1,\theta) = \bar{p}$$
$$u_P^+(m,\theta) = 0, \quad u_P^-(1,\theta) = -\bar{p}$$

The upper boundary set $\hat{bd}(Z)$ from Z is the set of allocations such that for some k they assign the buyer the goods with the smallest $(k-1)$th index for sure and good k for some probability:

$$X := \hat{bd}(Z) = \{z \in Z | \exists\ k \text{ s.t. } z^+(m) = 0 \text{ if } m > k \text{ and } 1 \text{ if } 1 \leq m < k\}.$$

Note that all the available bundles in X are totally ordered with the order $\succeq_{bd}$ on $\hat{bd}(Z)$ given in (5), which is the standard component-wise order on the set of bundles. Hence, according to the definition of upgrade pricing, the optimal selling mechanism under X is upgrade pricing. Therefore, the sufficient condition for the optimality of upgrade pricing can be obtained by finding when (Θ, Z, U) is single-crossing-reducible under X. Due to Theorem 2, the condition will turn out below to be type-monotonicity of valuation for each good and MRS between every pair of goods. Moreover, by using the linear payoff functions of the players with the single crossing condition of (Θ, X, U), the same intuition for the optimality of posting a monopoly price in the single good setting leads to the optimality of upgrade pricing with the menu of

[22] In the case of additive valuation, because for every good $l \in \{1, \cdots, d\}$ $v_\theta(l)$ is nonnegative, it always holds that for every bundle $b \in B$ $v_\theta(b^*) \geq v_\theta(b)$.

deterministic bundles from $\{[1],[2],\cdots,[d]\} \subseteq X^*$ where for $1 \le k \le d$ $[k] := \{1,\cdots,k\}$ is the bundle containing all the goods with index lower than equal to k. In addition, for each deterministic upgrade pricing, type-monotone valuations enable us to construct an equivalent separate pricing in terms of the allocation to each type and the prices for the bundles used by the upgrade pricing. Therefore, separate sales is also optimal.

Corollary 2 *Upgrade pricing is optimal for all distribution $\mu \in \Delta\Theta$ if there exists a total order on Θ and a permutation σ of the set of goods $\{1,\cdots,d\}$ such that for every θ,θ' with $\theta' \succeq_\Theta \theta$ and m,m' with $m' \ge m$,*

$$\textit{Monotone valuation: } v_{\theta'}(m) \ge v_\theta(m)\ \forall 1 \le m \le d$$
$$\textit{Monotone MRS: } \frac{v_{\theta'}(\sigma(m'))}{v_{\theta'}(\sigma(m))} \ge \frac{v_\theta(\sigma(m'))}{v_\theta(\sigma(m))}$$

Furthermore, there exists an optimal mechanism whose allocation is deterministic and belongs to $\{[1]_\sigma,[2]_\sigma,\cdots,[d]_\sigma\}$. Moreover, separate monopoly pricing is also optimal.[23]

It is worth noting the difference between our condition and those of Bergemann, Bonatti, Haupt, and Smolin (2021)(henceforth, BBHS). First, as already emphasized in the proposition, unlike those of BBHS, my condition (*monotone valuation* and *monotone MRS*) is sufficient for the optimality of upgrade pricing irrespective of type distribution $\mu \in \Delta\Theta$. Conditions from BBHS are dependent on the type distribution μ because they require that the type distribution is "mostly regular" so that they can sequentially apply on ironing procedure for each good.[24] Therefore, there can exist some Θ such that my condition fails, but for some μ the conditions of BBHS hold.[25]

However, for some Θ satisfying our condition, there can exist a type distribution that does not satisfy the regularity conditions of BBHS. The conditions of BBHS are to reduce the problem to d separate single good problems, which necessitates some regularity for each good. In contrast, under my condition the problem is reduced to a single crossing screening problem under the allocation subset X. Therefore, at least under X, regularity is not required for the

[23] Yang (2021) derives another sufficient condition for the optimality of upgrade pricing when the buyer's valuation function is non-additive.

[24] Roughly, they require that candidate ironing intervals for each item k, the interval under which the monotonicity of virtual valuation is violated, are almost disjoint between different goods.

[25] For example, in the case when the type distribution is mostly regular, BBHS requires only a weak monotonicity of valuations that requires order comparisons only between a pair of types above and below a cutoff, rather than every pair of types.

optimality of upgrade pricing if only a single crossing property holds on the buyers' preferences, which is equivalent to *monotone valuation*.

The condition is a necessary condition for the optimality of upgrade pricing in some cases. That is, when the type set has *strictly monotone valuation*, which means that valuation for each good is strictly increasing in type, monotone MRS becomes necessary and sufficient for the robust optimality of upgrade pricing with respect to type distribution.

As in the example, this can be shown by finding a type distribution for which upgrade pricing is not optimal. When valuation for each good is strictly increasing in type, for every two point distribution optimal mechanism would have the form of upgrade pricing. So, I need to find a three point distribution. There are two cases: If valuation for each good is sufficiently convex in type, I can construct a three point distribution for which optimal IC mechanism has only the local downward IC constraints binding, but it is not an upgrade pricing. If not, it has all the downward IC constraints binding, which also means that upgrade pricing is not optimal due to the strictly monotone valuation or the resulting strict single crossing property.

Proposition 6 *Suppose that the type space Θ has strictly monotone valuation with a total order $\succeq_\Theta$,* i.e.*, for every $k = 1, \cdots, d$, $v_\theta(k)$ is strictly increasing according to the type order $\succeq_\Theta$. Then, upgrade pricing is* ***robustly*** *optimal with respect to type distribution iff MRS between every pair of goods is monotone according to $\succeq_\Theta$.*

6.2 Bayesian persuasion

I now consider the persuasion problem studied by Guo and Shmaya (2019) where the receiver has private information about the state. As in their paper, I derive a sufficient condition for the optimality of *nested interval structure of acceptance sets*: each type's acceptance set is an interval and for every two different types, one's acceptance set is a subset of the other's. Relative to Guo and Shmaya (2019), it allows for more general form of sender's payoff function, as explained below.

Setup

There are two players: the sender and the receiver. The state set S is given as a bounded interval on the real line containing 0 with Lebesgue measure. The receiver privately knows his own type $\theta \in \Theta \subseteq \mathbb{N}$ where Θ is a finite set with the minimal element $\underline{\theta}$ and the maximum $\bar{\theta}$. The full support distribution over $S \times \Theta$ is given by a density $f(s, \theta)$ that is bounded and continuous in s.

At the start of the game, the sender commits to a disclosure mechanism (Y, κ, r) about S where κ is a Markov kernel from S to signal set Y and $r : Y \times \Theta \to \{0, 1\}$ is recommendation function where 0 means reject and 1 means accept. After observing the signal, the receiver chooses to accept or reject. At state $s \in S$, the receiver's payoff from accepting at state s is given by a bounded function $u(s) : S \to \mathbb{R}$ such that $u(0) = 0$, nonzero otherwise and $u(s) > 0$ if and only if $s > 0$. The sender's payoff from the agent's accepting at state s is also given by a bounded function $v(s) : S \to \mathbb{R}$ while his payoff from rejection at each state is zero. I assume that at every state the sender has a stronger preference for accepting over rejecting than the receiver: if for some s $u(s) \geq 0$ then $v(s) > 0$. That is, there exists some state $s_0 \leq 0$ such that $v(s) > 0$ if and only if $s > s_0$.[26] Note that in Guo and Shmaya (2019) the receiver's payoff from accepting is additionally nondecreasing in s while the principal's payoff from accepting is always nonnegative for every state, which definitely satisfies our assumption.

Nested interval structure of acceptance sets

To fit this problem into our setting, I consider a more relaxed problem. Given a disclosure mechanism (Y, κ, r), one can calculate the acceptance probability at each state s when the agent always follows the recommendation r. It is denoted by

$$h(s, \theta) \equiv \int r(y, \theta)\kappa(s, dy)$$

where $h(\theta) \equiv h(\cdot, \theta) : S \to [0, 1]$ is called an *acceptance set*. Let Z be the set of all such acceptance sets:

$$Z = \{z = (z^+, z^-) \in \mathbb{R}_+^{C^+} \times \mathbb{R}_+^{C^-} | \mathbb{1} \succeq_0 z\}$$

where $C^+ = S \cap [0, \infty)$ and $C^- = \{-s | s \in S \cap (-\infty, 0]\}$.

Then, the principal and the agent's utilities for each type θ are determined by the acceptance set $h(\theta)$. Because I assume the common prior between the players as in Guo and Shmaya (2019), the players' expected utilities are

$$U_A(h(\theta), \theta) = \int_{c \in S^+} h(c, \theta) u_A^+(c, \theta) dc - \int_{c \in S^-} h(-c, \theta) u_A^-(c, \theta) dc$$
$$U_P(h(\theta), \theta) = \int_{c \in S^+} h(c, \theta) u_P^+(c, \theta) dc - \int_{c \in S^-} h(-c, \theta) u_P^-(c, \theta) dc$$

[26] This is without loss of generality because unlike Guo and Shmaya (2019) we do not assume that the receiver's payoff u is nondecreasing in s.

where $u_A^+(c,\theta) := f(c,\theta)u(c)$ and $u_P^+(c,\theta) := f(c,\theta)v(c)$ for every $c \in C^+$ and $u_A^-(c,\theta) := -f(-c,\theta)u(-c)$ and $u_P^-(c,\theta) := -f(-c,\theta)v(-c)$ for every $c \in C^-$.[27]

Then, upper boundary set $\hat{bd}(Z)$ of Z is the set of all intervals $[-a,b]$ for $a,b \geq 0$, which can be also defined as

$$X = \hat{bd}(Z) = \{(\mathbb{1}_{[0,b]}, \mathbb{1}_{[0,a]}) | a,b \geq 0\}.$$

Here, I will use both notations, $[-a,b]$ and $(\mathbb{1}_{[0,b]}, \mathbb{1}_{[0,a]})$, equivalently. The corresponding order $\succeq_{bd}$ on $\hat{bd}(Z)$ is also defined by that

$$[-a',b'] \succeq_{bd} [-a,b] \ \Leftrightarrow \ b' \geq b,$$

If (Θ, X, U) is a single crossing problem with $\succeq_{bd}$, optimal IC mechanism h^* under X takes the form of nested structure of acceptance sets: for every θ and θ', $h^*(\theta) \subseteq h^*(\theta)$ or vice versa. Hence, for the optimality of nested interval structure of acceptance set for (Θ, Z, U), it suffices to show that (Θ, Z, U) is single-crossing-reducible under X.

Compared to the original problem, the screening problem (Θ, Z, U) where the sender offers an acceptance probability is more relaxed in terms of the set of feasible acceptance probability functions. More specifically, in (Θ, Z, U) when type θ agent deviates under a mechanism h, he has to choose an acceptance set $h(\theta')$ for some $\theta' \neq \theta$. However, in the original problem type θ agent can choose not only $h(\theta')$ but also every acceptance set that can be generated by choosing whether to accept or reject for each signal. For example, each type of agent can choose every acceptance set that can be gained by the set operation between the elements of $\{h(\Theta) | \theta \in \Theta\}$. However, if an optimal IC mechanism h^* under Z has a nested interval structure of acceptance sets due to the single-crossing-reducibility of (Θ, Z, U) under X, the single crossing property of the agent's preference over X guarantees that such possible deviations would also not be profitable, and thus, h^* is optimal in the original problem.

Thus, nested interval structure of acceptance sets is optimal if (Θ, Z, U) can be reduced to a single crossing problem under X. However, Theorem 2 cannot be applied in this case because the increasing MRS condition and the increasing conflict of interest condition might conflict with each other. This is due to the common prior belief between the principal and the agent.[28] For example, when $s_0 = -\infty$ as in Guo and Shmaya (2019), since $\frac{u_A^+(s',\theta')}{u_A^-(s,\theta')} / \frac{u_A^+(s',\theta)}{u_A^-(s,\theta)} =$

[27] To simplify the exposition, here I multiply the players' expected utilities $\mathbb{E}_s[h(s,\theta)u(s)|\theta]$ and $\mathbb{E}_s[h(s,\theta)v(s)|\theta]$ for each type θ by the marginal probability $\int f(s,\theta)ds$. This is without loss of generality for applying my result that is robust to distribution over the type set.

[28] In more general setup that assumes heterogeneous beliefs as in Section 5.1 of Guo and Shmaya (2019), both condition can hold together.

$\frac{u_P^+(s',\theta')}{u_P^-(s,\theta')}/\frac{u_P^+(s',\theta)}{u_P^-(s,\theta)}$ for every θ and $s \leq 0 \leq s'$, type-increasing $\frac{u_A^+(s',\theta)}{u_A^-(s,\theta)}$ (Definition 7) and type-decreasing $-\frac{u_P^+(s',\theta)}{u_P^-(s,\theta)}$ (Proposition 4) cannot hold together.

Hence, I weaken the increasing conflict of interest condition so that it still guarantees the equivalence between optimal IC and LIC mechanisms under X in the following proposition. See Appendix G for a weaker version of increasing conflict of interest in the general setting, as well as the relevant proofs.

Proposition 7 *Suppose that the type set Θ has increasing MRS. Then, optimal IC and LIC mechanisms under $X = \widehat{bd}(Z)$ give the principal the same expected payoff if $\frac{v(s)}{u(s)} \leq \frac{v(s')}{u(s')}$ for every $s \leq s_0 \leq 0 \leq s'$.*

By using this proposition with Theorem 2, a sufficient condition for the reduction to a single crossing problem can be obtained.

Corollary 3 *An optimal mechanism has a form of nested interval structure of acceptance sets if the following conditions hold together:*

$$\text{for every } s' \geq s,\ \frac{f(s',\theta)}{f(s,\theta)} \text{ is nondecreasing in type} \tag{8}$$

$$\text{for } s \leq s' < 0 \text{ and } 0 < s \leq s',\ \ \frac{v(s)}{u(s)} \geq \frac{v(s')}{u(s')} \tag{9}$$

$$\text{for } s \leq s_0 \leq 0 \leq s',\ \ \frac{v(s)}{u(s)} \leq \frac{v(s')}{u(s')}. \tag{10}$$

It is worth noting that if as in Guo and Shmaya (2019) the receiver's payoff $u(s)$ is nondecreasing in s and the sender's payoff $v(s)$ is positively constant of s (that is, $s_0 = -\infty$), our optimality result holds if only (8) holds, which is equivalent to the *i.m.l.r* assumption made in Guo and Shmaya (2019). Compared to Guo and Shmaya (2019), a stronger result about the equivalence between the optimal LIC and IC mechanisms under the allocation subset, which is Proposition 7, allows us to extend their result to a persuasion problem with more general form of the sender's payoff function.[29]

[29] In Guo and Shmaya (2019), the equivalence result is shown by a construction of an optimal IC mechanism from an optimal LIC mechanism through expanding acceptance sets for higher types. This process can be done with keeping the sender's expected payoff being increased only when the sender always prefers the receiver's accepting to rejecting.

6.3 Optimal Delegation

In this section, I study a delegation problem where the agent's private bias is constant and one-sided. Applying my result, I derive sufficient conditions for the optimality of no screening with offering the same cap to the agent regardless of his bias.

Setup

The principal has the legal right to make the decision, but only the expert is knowledgeable to make a right decision depending on the true state $s \in [0, 1]$. That is, the expert perfectly observes s before making an action choice while the principal only knows its prior distribution G with support $[0, 1]$ and with density g that is absolutely continuous and strictly positive for all $s \in S = [0, 1]$.

The principal and the agent's utilities depend on the implemented decision $a \in \mathbb{R}$, the state s, and the agent's type or bias $\theta \in \Theta$ where the type set Θ is finite and only has nonnegative elements. The bias $\theta \geq 0$ is unknown to the principal who only has a prior belief μ about the type. I consider a function $u : \mathbb{R} \to \mathbb{R}$ that is symmetric, differentiable, strictly concave, and maximized at 0. The principal's payoff function is defined as

$$u_P(a, s) = u(a - s),$$

while the payoff function of the agent with type θ:

$$u_A^\theta(a, s) = u(a - \theta - s).$$

Before the state is realized, the principal selects and commits to a direct mechanism $h : \Theta \to Z$ where Z is a set of direct mechanisms played after the state realization. Each direct mechanism $z \in Z$ maps from the state set S to the set of admissible actions $\mathbb{R}$. Note that here I restrict our attention to the deterministic mechanism where offering a lottery over different actions is not allowed as in Alonso and Matouschek (2008). Thus, each allocation $z \in Z$ is equivalent to a delegation set from which the agent chooses his preferred one.

For each type θ, the agent θ's payoff from a delegation set $z \in Z$, denoted as $U_A(z, \theta)$, is equal to

$$U_A(z, \theta) = \int_0^1 u_A^\theta(\tilde{a}(s, z, \theta), s) g(s) ds$$

where $\tilde{a}(s, z, \theta)$ is the agent θ's optimal action when the state is s and the delegation set z is given. The corresponding principal's payoff from offering the delegation set z to the agent θ, denoted as $U_P(z,\theta)$, is equal to

$$U_P(z,\theta) = \int_0^1 u_P(\tilde{a}(s,z,\theta), s)g(s)ds.$$

Then, the problem becomes equivalent to the screening problem (Θ, Z, U) without the IR constraints.[30]

No screening with the same cap

To apply my result in this section, I need to reformulate the allocation set and the payoff functions. First, it is without loss of generality to restrict the attention to the delegation set that contains $(-\infty, \underline{c}]$ for sufficient small $\underline{c} < 0$.[31] Because the support of the state set is $[0,1]$, for each type θ it is without loss to consider an extended state set $[\underline{c} - \theta, \infty]$.

Next, I need the following lemma about the agent's optimal action choice under a delegation set.[32] I define $\tilde{a}^-(s_0,z,\theta) = \lim_{s\to s_0^-} \tilde{a}(s,z,\theta)$ and $\tilde{a}^+(s_0,z,\theta) = \lim_{s\to s_0^+} \tilde{a}(s,z,\theta)$.

Lemma 2 *Suppose that a delegation set $z \subseteq \mathbb{R}$ that contains $(-\infty, \underline{c}]$ is given to an agent with bias θ. Then, the agent's optimal action $\tilde{a}(s,z,\theta) : [\underline{c} - \theta, \infty) \to \mathbb{R}$ satisfies the following condition.*

1. *$\tilde{a}(s,z,\theta)$ is nondecreasing in s.*
2. *If $\tilde{a}(s,z,\theta)$ is strictly increasing and continuous on an open interval (s_1,s_2), then $\tilde{a}(s,z,\theta) = s+\theta$ on (s_1,s_2).*
3. *If $\tilde{a}(s,z,\theta)$ is discontinuous at s_0 for some $\theta \in \Theta$, the discontinuity must be a jump discontinuity that satisfies*
   - $u_A^\theta(\tilde{a}^-(s_0,z,\theta), s_0) = u_A^\theta(\tilde{a}^+(s_0,z,\theta), s_0)$, i.e., $s_0 + \theta = \frac{\tilde{a}^-(s_0,z,\theta)+\tilde{a}^+(s_0,z,\theta)}{2}$
   - *$\tilde{a}(s,z,\theta) = \tilde{a}^-(s_0,z,\theta)$ for $s \in [\tilde{a}^-(s_0,z,\theta) - \theta, s_0)$ and $\tilde{a}(s,z,\theta) = \tilde{a}^+(s_0,z,\theta)$ for $s \in (s_0, \tilde{a}^+(s_0,z,\theta) - \theta]$.*
   - *$\tilde{a}(s_0,z,\theta) \in \{\tilde{a}^-(s_0,z,\theta), \tilde{a}^+(s_0,z,\theta)\}$.*

[30] It is straightforward to show that my result still holds even when the IR constraints are not required.

[31] When $\underline{c}$ is sufficiently small, all types of the agent will not change their actions under the optimal mechanism even if they are allowed to choose actions from $(-\infty, \underline{c}]$ as well as the delegation sets.

[32] I omit the proof here because it is equivalent to Proposition 1 in Melumad and Shibano (1991) or Lemma 2 in Alonso and Matouschek (2008).

4. *If $\tilde{a}(s,z,\theta)$ is discontinuous at s_0 for some θ, then for each $\theta' \in \Theta$ $\tilde{a}(s,z,\theta')$ is discontinuous at $s_0 - \theta + \theta'$.*

Because $\tilde{a}(s,z,\theta)$ only has countable discontinuities, without loss one can assume that it is right-continuous. When $\tilde{a}(s-\theta,z,\theta) = \underline{c} + \int_{\underline{c}}^{s} d\tilde{a}(c-\theta,z,\theta)$, from the above lemma the following type independence and dominance relation is directly obtained.

Proposition 8 *For every delegation set z, $\tilde{a}(s,z,\theta)$ satisfies the following:*

- *$\tilde{a}(s-\theta,z,\theta) : [\underline{c},\infty) \to \mathbb{R}$ is independent of θ.*
- *$\mathbb{1} \succeq_2 d\tilde{a}(s-\theta,z,\theta)$ when both are over $C^+ := [\underline{c},\infty)$.*

Due to the first condition in Proposition 8, the allocation set Z can be redefined as the set of all the derivatives of optimal action choice function with $C^+ = [\underline{c},\infty)$, $C^- = \emptyset$, and

$$Z = \{d\tilde{a}(s-\theta,\tilde{z},\theta) : [\underline{c},\infty) \to \mathbb{R}_+ | (-\infty,\bar{c}] \subseteq \tilde{z} \subseteq \mathbb{R}\}.$$

Also, due to the second condition of Proposition 8, Z is a subset of $\bar{Z} := \{z : [\underline{c},\infty) \to \mathbb{R}_+ | \mathbb{1} \succeq_2 z\}$.

For the rest of this section, I assume that the loss function u is quadratic, *i.e.*, $u(x) = -x^2$. Then, for the allocation set Z, the payoff function U can be also redefined by using integration by parts as in Alonso and Matouschek (2008):

$$U_A(z,\theta) = U_A((-\infty,\underline{c}],\theta) + \int_{\underline{c}}^{\infty} u_A^+(c^+,\theta)\cdot z(c^+)dc^+$$
$$U_P(z,\theta) = U_P((-\infty,\underline{c}],\theta) + \int_{\underline{c}}^{\infty} u_P^+(c^+,\theta)\cdot z(c^+)dc^+$$

where the players' marginal benefits from allowing to choose c are given as the followings:

$$u_A^+(c^+,\theta) = (1-G(c^+-\theta))(\mathbb{E}[\tilde{c}|\tilde{c} \geq c^+-\theta] - (c^+-\theta))$$
$$u_P^+(c^+,\theta) = (1-G(c^+-\theta))(\mathbb{E}[\tilde{c}|\tilde{c} \geq c^+-\theta] - c^+).$$

Then, upper boundary set $\hat{bd}(\bar{Z})$ of $\bar{Z}$ is

$$\hat{bd}(\bar{Z}) := \{\mathbb{1}_{[\underline{c},c']} | c' \geq \underline{c}\},$$

and thus, the corresponding set of the agent's optimal action choices is the set of choice functions for every cap $(-\infty,c']$. Also, because $\hat{bd}(\bar{Z}) \subseteq Z$, $\hat{bd}(Z) = \hat{bd}(\bar{Z})$. Therefore, if the screening problem (Θ, Z, U) is single-crossing-reducible under $X := \hat{bd}(Z)$, this implies that offering each type of the agent a cap is optimal in the delegation problem (Θ, Z, U). Moreover, because all the agents prefer higher caps and do so more strongly than the principal with any type of the agent, the optimal IC mechanism offers the same cap regardless of the agent's type. Thus, from Theorem 2, it can be shown that the increasing hazard ratio of g is a sufficient condition for no screening with the same cap.

Corollary 4 *Suppose that the loss function $u(x) \equiv -x^2$. Then, offering same cutoff to every type of the agent is optimal if $\frac{g}{1-G}(s)$ is nondecreasing in s.*

Remark 3 *Under the uniform state distribution and the power loss function, Tanner (2018) finds another sufficient condition for no screening with the same cap, which is that the players are sufficiently risk averse. By using Theorem 1, one can not only reproduce his condition but also extend it to more general payoff functions.*[33]

7 Conclusion

In this paper, I provided a sufficient condition about the agent's type set under which the principal can restrict attention, without profit loss, to a subset of the allocations over which the agent's preference has the single crossing property. By using the result, I have derived a sufficient condition for the optimality of *nested mechanisms* where all the available allocations are ordered with the standard componentwise order.

The results can be applied to various screening problems with multiple allocations to find sufficient conditions for the optimality of simple mechanisms. For example, in the multiple good monopoly problem, upgrade pricing is optimal if for a total order on the types the valuation for each good is increasing and the MRS between every pair of goods is monotone. Monotone MRS also becomes a necessary condition for its optimality when the valuation for each good is strictly increasing. In the delegation problem with a one-sided and constant bias and quadratic loss function, no screening offering a single cap is optimal when the hazard rate of the state distribution is increasing.

For future research, more results about the optimality of simple mechanisms might be obtained by using my result in other screening problems such as taxation, auctions, and insurance.

[33]When the state distribution g is uniform, offering same cutoff to every type of the agent is optimal if u' is concave, and $\frac{u'}{u}(x)$ is nonincreasing in x when $x < 0$. This condition about u is satisfied by the power loss function $u(x) = -|x|^l$ for $l \geq 2$ that is assumed in Tanner (2018). See Online Appendix.

Appendix

A Proof of the main result

A.1 Proof of Proposition 1

I prove this by showing that for every LIC mechanism h under Z with $\succeq_\Theta$, there exists an LIC mechanism h_X under X with $\succeq_\Theta$ that gives the principal at least the same expected payoff as h does.

Consider an LIC mechanism h under Z with $\succeq_\Theta$, which means that for every $\theta,\ \theta' \in \Theta$ with $\theta' \succeq_\Theta \theta$,

$$U_A(h(\theta'), \theta') \geq U_A(h(\theta), \theta') \tag{11}$$

$$U_A(h(\theta), \theta) \geq 0 \tag{12}$$

Then, because of the definition of incentive contract curve, there exists a function $\pi : Z \times \Theta \to X$ s.t. for each $\theta \in \Theta$ and $h(\theta)$, $\pi(h(\theta), \theta)$ satisfies the followings:

$$U_A(\pi(h(\theta), \theta), \theta) \geq U_A(h(\theta), \theta) \tag{13}$$

$$U_P(\pi(h(\theta), \theta), \theta) \geq U_P(h(\theta), \theta) \tag{14}$$

$$U_A(\pi(h(\theta), \theta), \theta') \leq U_A(h(\theta), \theta') \text{ for } \theta' \text{ s.t. } \theta' \succeq_\Theta \theta \tag{15}$$

Now, I show that the mechanism $h_X : \Theta \to X$ consisting of $h_X(\theta) := \pi(h(\theta), \theta)$ for each $\theta \in \Theta$ satisfies all the IR and downward IC constraints, and it gives at least the same payoff to the principal as h.

(a) IR

As (12) says, h satisfies all the IR constraints. According to (13), for each θ $h_X(\theta)$ gives more payoff to the agent than $h(\theta)$ does, and thus, h_X also satisfies all the IR constraints.

(b) Downward IC

This is because for all $\theta,\ \theta'$ s.t. $\theta' \succeq_\Theta \theta$

$$U_A(\pi(h(\theta'), \theta'), \theta') \geq U_A(h(\theta'), \theta') \geq U_A(h(\theta), \theta') \geq U_A(\pi(h(\theta), \theta), \theta')$$

where the first, second, and third inequalities are from (13), (11) and (15), respectively.

(c) Larger payoff under h_X than under h

This is because of (14) which implies that for each θ the principal will get a higher payoff from $h_X(\theta)$ than $h(\theta)$.

A.2 Proof of Proposition 2

For the proof, I focus on optimal LIC mechanisms under X that satisfy a property called *agent-optimality* defined below. For the ease of exposition, I denote as θ_n the nth lowest type according to the total order $\succeq_\Theta$ for the single crossing condition of (Θ, X, U). That is, $\Theta = \{\theta_1, \theta_2, \cdots, \theta_N\}$ for some $N > 0$ and $\theta_N \succeq_\Theta \cdots \succeq_\Theta \theta_2 \succeq_\Theta \theta_1$.

Definition 9 *An optimal LIC mechanism h under X with $\succeq_\Theta$ is **agent-optimal** if there does not exist another optimal LIC mechanism h' under X with $\succeq_\Theta$ such that for every θ,*

$$U_A(h'(\theta), \theta) \geq U_A(h(\theta), \theta),$$

and either

- *for some $\theta \in \Theta$ such that $U_A(h'(\theta), \theta) > U_A(h(\theta), \theta)$, or*
- $\sum_{n=1}^{N-1} n\mathbb{1}_{h'(\theta_n)=h'(\theta_{n+1})} > \sum_{n=1}^{N-1} n\mathbb{1}_{h(\theta_n)=h(\theta_{n+1})}$.

The definition says that an optimal LIC mechanism h under X is agent-optimal if there does not exist another optimal LIC mechanism under X such that it is weakly preferred to h by all types of the agent and either it is strictly preferred by at least one type or the variance among the allocations to the highest type agents is smaller.[34]

Then, the following Lemma, which shows that in a single crossing problem with the property of increasing conflict of interest every agent-optimal optimal LIC mechanism is an optimal IC mechanism, implies Proposition 2.

Lemma 3 *If with $(\succeq_X, \succeq_\Theta)$ (Θ, X, U) is a single crossing screening problem and has the property of increasing conflict of interest, every agent-optimal optimal LIC mechanism under X with $\succeq_\Theta$ is an optimal IC mechanism under X.*

Proof. Suppose not. That is, given $(\succeq_X, \succeq_\Theta)$ with which (Θ, X, U) is a single crossing one and the property of increasing conflict of interest holds, there exists an agent-optimal optimal LIC mechanism h under X which is not an optimal IC mechanism under X, *i.e.*, h does not satisfy an upward IC constraint for a pair of types. Moreover, because of the single crossing property of the agent's preference over X, it is without loss to assume that a local upward IC

[34] The second condition can be ruled out if the agent's preference over X has a strict single crossing property

constraint between the adjacent types θ_n and θ_{n+1} for some $1 \leq n \leq N-1$ does not hold. Let n^* be the minimum among such n, which means that

$$U_A(h(\theta_{n^*+1}), \theta_{n^*}) > U_A(h(\theta_{n^*}), \theta_{n^*}), \tag{16}$$

and for $n < n^*$,

$$U_A(h(\theta_{n+1}), \theta_n) \leq U_A(h(\theta_n), \theta_n). \tag{17}$$

Step 1 Because h is agent-optimal, for every $n < n^*$, at least one of upward or downward IC constraints between θ_{n+1} and θ_n is non-binding unless $h(\theta_{n+1}) = h(\theta_n)$, which implies that $h(\theta_{n+1}) \succeq_X h(\theta_n)$.

Proof. Suppose not. That is, for $h(\theta_{n+1}) \neq h(\theta_n)$,

$$(U_A(h(\theta_{n+1}), \theta_{n+1}), U_A(h(\theta_{n+1}), \theta_n)) = (U_A(h(\theta_n), \theta_{n+1}), U_A(h(\theta_n), \theta_n)). \tag{18}$$

Then, one can construct two LIC mechanisms h_1^n and h_2^n by replacing $h(\theta_n)$ with $h(\theta_{n+1})$ and vice versa, respectively:

$$\begin{aligned} h_1^n &\equiv (h(\theta_1), \cdots, h(\theta_{n-1}), h(\theta_n), h(\theta_n), h(\theta_{n+2}), \cdots, h(\theta_N)) \\ h_2^n &\equiv (h(\theta_1), \cdots, h(\theta_{n-1}), h(\theta_{n+1}), h(\theta_{n+1}), h(\theta_{n+2}), \cdots, h(\theta_N)). \end{aligned}$$

Because h is an optimal LIC mechanism, the principal has to prefer h to h_1^n and h_2^n:

$$U_P(h(\theta_{n+1}), \theta_{n+1}) \geq U_P(h(\theta_n), \theta_{n+1}) \tag{19}$$

$$U_P(h(\theta_n), \theta_n) \geq U_P(h(\theta_{n+1}), \theta_n). \tag{20}$$

Then, the property of increasing conflict of interest with (18) and (19) implies that

$$U_P(h(\theta_n), \theta_n) \leq U_P(h(\theta_{n+1}), \theta_n),$$

which with (20) also implies that

$$U_P(h(\theta_n), \theta_n) = U_P(h(\theta_{n+1}), \theta_n).$$

Combined with (18), this equation means that for h, $h' = h_2^n$ satisfies the second condition in Definition 9, which contradicts with the agent-optimality of h. $\square$

Then, I construct two mechanisms $h_1^{n^*}$ and $h_2^{n^*}$ by replacing $h(\theta_{n^*})$ with $h(\theta_{n^*+1})$ and vice versa, respectively:

$$\begin{aligned}
h_1^{n^*} &\equiv (h(\theta_1), \cdots, h(\theta_{n^*-1}), h(\theta_{n^*}), h(\theta_{n^*}), h(\theta_{n^*+2}), \cdots, h(\theta_N)) \\
h_2^{n^*} &\equiv (h(\theta_1), \cdots, h(\theta_{n^*-1}), h(\theta_{n^*+1}), h(\theta_{n^*+1}), h(\theta_{n^*+2}), \cdots, h(\theta_N)).
\end{aligned}$$

Step 2 Both $h_1^{n^*}$ and $h_2^{n^*}$ are LIC mechanisms under X.

Proof. It is straightforward to show that $h_2^{n^*}$ is a LIC mechanism because of (16). For $h_1^{n^*}$, it suffices to show that for $n \leq n^*$, the agent with type θ_{n^*+1} prefers $h(\theta_{n^*})$ to $h(\theta_n)$ and the outside option:

$$U_A(h(\theta_{n^*}), \theta_{n^*+1}) \geq U_A(h(\theta_n), \theta_{n^*+1}) \geq 0. \tag{21}$$

From Step 1, $h(\theta_{n'}) \succeq_X h(\theta_{n'-1})$ for every $n' \leq n^*$. Thus, due to the single crossing property of U_A, the downward IC between type $\theta_{n'}$ and $\theta_{n'-1}$ implies that the agent with type θ_{n^*+1} also prefers $h(\theta_{n'})$ to $h(\theta_{n'-1})$ while the IR for type $\theta_{n'-1}$ implies that the agent with type θ_{n^*+1} also prefers $h(\theta_{n'-1})$ to the outside option:

$$U_A(h(\theta_{n'}), \theta_{n^*+1}) \geq U_A(h(\theta_{n'-1}), \theta_{n^*+1}) \geq 0. \tag{22}$$

Hence, by combining (22) for $n' = n+1, \cdots, n^*$, (21) can be obtained. □

Step 3 Because h is an optimal LIC mechanism under X, the principal weakly prefers h to $h_1^{n^*}$:

$$U_P(h(\theta_{n^*+1}), \theta_{n^*+1}) \geq U_P(h(\theta_{n^*}), \theta_{n^*+1}).$$

Because h is agent optimal and (16) holds, the principal strictly prefers h to $h_2^{n^*}$:

$$U_P(h(\theta_{n^*}), \theta_{n^*}) > U_P(h(\theta_{n^*+1}), \theta_{n^*}).$$

These two inequalities with (16) and the downward IC between θ_{n^*} and θ_{n^*+1}

$$U_A(h(\theta_{n^*+1}), \theta_{n^*+1}) \geq U_A(h(\theta_{n^*}), \theta_{n^*+1}) \tag{23}$$

cannot hold together due to the property of increasing conflict of interest, which is a contradiction. ■

A.3 Proof of Theorem 1

The first condition of the reducibility to a single crossing problem under X holds (Θ, X, U) is a single crossing problem. Hence, it suffices to show the second condition, the equivalence between optimal IC mechanism under Z and X. According to Proposition 2, if (Θ, X, U) is a single crossing problem with the property of increasing conflict of interest, every optimal IC mechanism under X is an optimal LIC mechanism under X. Also, according to Proposition 1, if X is an incentive contract curve, every optimal LIC mechanism under X is an optimal LIC mechanism under Z. Therefore, every optimal IC mechanism under X is an optimal LIC mechanism under Z. Because every optimal LIC mechanism under Z that satisfies all the ICs has to be an optimal IC mechanism under Z, every optimal IC mechanism under X is an optimal IC mechanism under Z, which completes the proof.

B Implementation

The main challenge in applying my result to a screening problem is finding an incentive contract curve small enough to reduce the problem to single crossing one. In this section, I present one way to construct the smallest incentive contract curve according to set inclusion order.[35]

We first define *incentive dominance order*, which is a strict partial order on the allocation set Z, so that the incentive contract curve is the set of *maximal allocations*, that is, allocations that do not have a higher allocation according to the order.

Definition 10 *Given a total order $\succeq_\Theta$ on the type set Θ, an incentive dominance order $\succ_{\theta,Z}$ for each type θ is a strict partial order on the allocation set Z that satisfies the following: For a pair of allocations $z,\ z' \in Z$, if $z \succ_{\theta,Z} z'$, the following inequalities about the players' payoffs hold with at least one strict inequality*

- *For type θ, z Pareto dominates by z':*

$$U_A(z,\theta) \geq U_A(z',\theta),\ \ U_P(z,\theta) \geq U_P(z',\theta)$$

- *All the types strictly higher than θ prefers z to z':*

$$U_A(z,\theta') \leq U_A(z',\theta')\ \ \forall \theta' \succ_\Theta \theta.$$[36]

[35] Of course, because the set inclusion order is a partial one, there can be multiple number of the smallest incentive contract curve.

[36] $\theta' \succ_\Theta \theta$ means that $\theta' \succeq_\Theta \theta$ but $\theta' \neq \theta$.

We call the set of maximal allocations according to $\succ_{\theta,Z}$ the ***incentive frontier for*** θ*, denoted as* Z^*_θ*. Also, the union of the incentive frontiers for all types,* $Z^*_{\succeq_\Theta} := \cup_{\theta\in\Theta} Z^*_\theta$*, is called the* ***incentive frontier****.*

In other words, $\succ_{\theta,Z}$ is the strict componentwise order when each allocation z is represented by a vector consisting of the players' payoffs from z and type θ, $(U_A(z,\theta), U_P(z,\theta))$, and the payoffs of the higher types of the agent from z, $\{-U_A(z,\theta')|\theta' \succ_\Theta \theta\}$. Because the incentive frontier Z^*_θ for θ is the set of maximal allocations according to the order, for every allocation $z \in Z$ and every type $\theta \in \Theta$, there exists an allocation $z' \in Z^*_\theta$ such that all the three conditions in the definition of incentive contract curve are satisfied when $\pi(z,\theta) = z'$. Hence, the *incentive frontier*, the union of the incentive frontiers for all types, is an incentive contract curve according to Definition 3.

Lemma 4 *Given a total order $\succeq_\Theta$ on the type set Θ, the incentive frontier $Z^*_{\succeq_\Theta}$ is an incentive contract curve.*

Proposition 9 *Every optimal downward IC mechanism under the incentive frontier $Z^*_{\succeq_\Theta}$ is an optimal downward IC mechanism under Z.*

Hence, we obtain the following result that directly comes from our main result.

Theorem 3 *Suppose that there exist a total order $\succeq_\Theta$ on Θ such that for a total order $\succeq^*$ on the incentive frontier $Z^*_{\succeq_\Theta}$, $(\Theta, Z^*_{\succeq_\Theta}, U)$ is a single crossing problem with increasing conflict of interest (Property (S). Then, for every type distribution $\mu \in \Delta\Theta$, every optimal IC mechanism under $Z^*_{\succeq_\Theta}$ is an optimal IC mechanism under Z.*

C The set of sufficient conditions for increasing conflict of interest

In this section, I provide a set of sufficient conditions for the property of increasing conflict of interest. The first sufficient condition is that the principal's preference over the allocation subset X is independent of the agent's type, *i.e.*, for every $x \in X$, $U_P(x,\theta)$ is constant for θ. An example that satisfies this condition is the standard single good monopoly where the principal maximizes the expected revenue. Combined with the single crossing condition shown in Example 1, Proposition 2 leads to the equivalence between optimal LIC and IC mechanisms in the single good monopoly.

The next sufficient condition is that the single crossing property of the agent' preferences U_A over X can be extended to that of the principal's preference U_P with satisfying some properties below. First, recall the definition of the single crossing property, which is introduced by Milgrom and Shannon (1994).

Definition 11 *(Milgrom and Shannon, 1994) Suppose that Y and Z are partially ordered set with $\succeq_Y$ and $\succeq_Z$, respectively. Then, $f : Y \times Z \to \mathbb{R}$ satisfies the **single crossing property** in (y, z) on $\tilde{Y} \times \tilde{Z} \subseteq Y \times Z$ with $(\succeq_Y, \succeq_Z)$ if for $y' \succeq_Y y''$ in $\tilde{Y}$ and $z' \succeq_Z z''$ in $\tilde{Z}$,*

$$f(y', z'') > f(y'', z'') \Rightarrow f(y', z') > f(y'', z')$$
$$f(y', z'') \geq f(y'', z'') \Rightarrow f(y', z') \geq f(y'', z')$$

In order to formalize the single crossing property of the players' preferences, we integrate $U_P|_{X\times\Theta}$ and $U_A|_{X\times\Theta}$ into $U|_{X\times\Theta} : X \times \Theta \times \{A, P\} \to \mathbb{R}$. Also, if the principal faces the agent with type $\theta \in \Theta$ denoted by $(\theta; A)$, then I say that the principal also has type θ and denote her by $(\theta; P)$. Of course, unlike the agent the principal does not know her type.

For defining the single crossing property of $U|_{X\times\Theta}$, one needs to choose orders on X and $\Theta \times \{A, P\}$. First, for the order on X, I keep using the total order $\succeq_X$ on X such that the agent's preference $U_A|_{X\times\Theta}$ satisfies the single crossing property with $(\succeq_X, \succeq_\Theta)$ as in Section 2.3. For an order $\succeq_{\Theta\times\{A,P\}}$ of the players' types $\Theta \times \{A, P\}$, I require it to satisfy the following two conditions together:

(O_1) For every $\theta, \theta' \in \Theta$ with $\theta \succeq_\Theta \theta'$,

$$(\theta; A) \succeq_{\Theta\times\{A,P\}} (\theta'; A).$$

(O_2) For every $\theta, \theta' \in \Theta$ with $\theta \succeq_\Theta \theta'$, $(\theta'; P)$ is between two of $(\theta; A)$, $(\theta; P)$, and $(\theta'; A)$.

The single crossing property of $U|_{X\times\Theta}$ with $(\succeq_X, \succeq_{\Theta\times\{A,P\}})$ becomes consistent with that of $U_A|_{X\times\Theta}$ with $(\succeq_X, \succeq_\Theta)$ due to (O_1) while it implies the property of increasing conflict of interest due to (O_2). Hence, from Proposition 2, I obtain the following corollary.

Corollary 5 *Suppose that a screening problem (Θ, X, U) is a single crossing one with a pair of total orders $(\succeq_X, \succeq_\Theta)$ on X and Θ and one of the followings holds:*

(S_1) The principal's preference U_P over X is independent of the agent's type, i.e., for every $x \in X$ and $\theta, \theta' \in \Theta$,

$$U_P(x, \theta) = U_P(x, \theta').$$

(S_2) *For every* $x, x' \in X$, $U|_{X\times\Theta} : X\times\Theta\times\{P, A\} \to \mathbb{R}$ *satisfies the single crossing property on* $\{x, x'\}\times\Theta\times\{P, A\}$ *with* $(\succeq_X, \succeq)$ *for some order* $\succeq_{\Theta\times\{A,P\}}$ *on the set of the players' types* $\Theta\times\{P, A\}$ *with the properties* (O_1) *and* (O_2).

Then, an optimal LIC mechanism under X *is an optimal IC mechanism under* X *for all type distributions* $\mu \in \Delta\Theta$.

The following two total orders on the set of players' types $\Theta \times \{P, A\}$, $\succeq^A$ and $\succeq^D$, are examples that satisfy the property (O_1) and (O_2). Hence, (S_2) holds if $U|_{X\times\Theta}$ satisfies the single crossing property with a pair of $\succeq_X$ and either of them.

$$\succeq^A_\Theta\text{:} \left\{ \begin{array}{c} (\theta; i) \succeq^A_\Theta (\theta'; i) \text{ for } i \in \{P, A\} \text{ and } \theta,\ \theta' \in \Theta \text{ with } \theta \succeq_\Theta \theta' \\ \\ (\theta; P) \succeq^A_\Theta (\theta'; A) \text{ for every } \theta,\ \theta' \in \Theta \end{array} \right\}$$

$$\succeq^D_\Theta\text{:} \left\{ \begin{array}{c} (\theta; A) \succeq^D_\Theta (\theta'; A) \text{ for every } \theta,\ \theta' \in \Theta \text{ with } \theta \succeq_\Theta n' \\ \\ (\theta'; P) \succeq^D_\Theta (\theta; P) \text{ for every } \theta,\ \theta' \in \Theta \text{ with } \theta \succeq_\Theta n' \\ \\ (\theta; A) \succeq^D_\Theta (\theta'; P) \text{ for every } \theta,\ \theta' \in \Theta \end{array} \right\}$$

It can be easily shown that both orders satisfy (O_1) and (O_2). Also, under $\succeq^A_\Theta$ every type of the principal is in a higher rank than every type of the agent while under $\succeq^D_\Theta$ every type of the principal is in a lower rank than every type of the agent. In addition, $\succeq^A_\Theta$ arranges the types of the principal according to the order $\succeq_\Theta$ while $\succeq^D_\Theta$ arranges the types of the principal in its reverse order.

D The optimality of no screening

In this section, I derive a sufficient conditions for the optimality of no screening, *i.e.*, the principal offers every type of the agent same allocation. From Corollary 5, the following sufficient condition ($\hat{S_2}$) , which is stronger than (S_2), guarantees the equivalence between optimal IC and LIC mechanisms under X. Moreover, an optimal IC mechanism under X gives the same allocation to every type of the agent, which means the optimality of no screening under X.

Proposition 10 *Suppose that* (Θ, X, U) *is a single crossing screening problem with a pair of total orders* $(\succeq_X, \succeq_\Theta)$ *on* X *and* Θ *with the following property* $(\hat{S}_2)$*:*

$(\hat{S}_2)$ $U|_{X\times\Theta} : X \times \Theta \times \{P, A\} \to \mathbb{R}$ *satisfies the single crossing property with* $(\succeq_X, \succeq_\Theta^D)$ *where the order* $\succeq_\Theta^D$ *on* $\Theta \times \{P, A\}$ *is given in Appendix C.*

Then, for all type distributions $\mu \in \Delta\Theta$*, every optimal IC mechanism under* X *is an optimal LIC mechanism under* X*. Furthermore, there exists an optimal IC mechanism* h^* *under* X *such that* h^* *is constant for* θ*.*

Proof. For the ease of exposition, I denote $\Theta = \{\theta_1, \theta_2, \cdots, \theta_N\}$ for some $N > 0$ and $\theta_N \succeq_\Theta \cdots \succeq_\Theta \theta_2 \succeq_\Theta \theta_1$. The first part comes directly from Corollary 10. For the second part, suppose that given a single crossing problem (Θ, X, U) with $(\succeq_X, \succeq_\Theta)$ and Property $(\hat{S}_2)$, every optimal IC mechanism under X is not constant for θ. That is, letting n^* be the maximum among n that satisfies $h(\theta_{n+1}) \neq h(\theta_n) = \cdots = h(\theta_1)$ for an optimal IC mechanism h under X, n^* is strictly smaller than N. Denote as h^* the corresponding optimal IC mechanism under X for n^*.

First, because h^* is an optimal IC mechanism under X, the upward and downward IC constraints between θ_{n^*+1} and θ_{n^*}:

$$U_A(h^*(\theta_{n^*+1}), \theta_{n^*+1}) \geq U_A(h^*(\theta_{n^*}), \theta_{n^*+1}) \tag{24}$$
$$U_A(h^*(\theta_{n^*}), \theta_{n^*}) \geq U_A(h^*(\theta_{n^*+1}), \theta_{n^*}) \tag{25}$$

Also, from Corollary 5, h^* is also an optimal LIC mechanism under X. Using these facts, I proceed the following steps.

Step 1. The principals with type θ_{n^*+1} and θ_{n^*} prefers their own allocation to the other:

$$U_P(h^*(\theta_{n^*+1}), \theta_{n^*+1}) \geq U_P(h^*(\theta_{n^*}), \theta_{n^*+1}) \tag{26}$$
$$U_P(h^*(\theta_{n^*}), \theta_{n^*}) \geq U_P(h^*(\theta_{n^*+1}), \theta_{n^*}) \tag{27}$$

Proof. Suppose not. First, suppose that $U_P(h^*(\theta_{n+1}), \theta_{n+1}) < U_P(h^*(\theta_n), \theta_{n+1})$. Then, due to the single crossing property of the agent's preference over X, the following mechanism

$$h_1^{n^*} \equiv (h^*(\theta_1), \cdots, h^*(\theta_{n^*-1}), h^*(\theta_{n^*}), h^*(\theta_{n^*}), h^*(\theta_{n^*+2}), \cdots, h^*(\theta_N))$$

satisfies all the downward IC and IR constraints as well as gives the principal a strictly higher payoff than h, which is a contradiction.

Next, suppose $U_P(h^*(\theta_{n^*}), \theta_{n^*}) < U_P(h^*(\theta_{n^*+1}), \theta_{n^*})$. Then, due to the single crossing property $(\hat{S}_2)$, this with (24) implies that

$$U_A(h^*(\theta_{n^*}), \theta_{n^*}) \leq U_A(h^*(\theta_{n^*+1}), \theta_{n^*}).$$

Hence, with (25), $U_A(h^*(\theta_{n^*}), \theta_{n^*}) = U_A(h^*(\theta_{n^*+1}), \theta_{n^*})$. Therefore, the following mechanism

$$h_2^{n^*} \equiv (h^*(\theta_1), \cdots, h^*(\theta_{n^*-1}), h^*(\theta_{n^*+1}), h^*(\theta_{n^*+1}), h^*(\theta_{n^*+2}), \cdots, h^*(\theta_N))$$

satisfies all the downward IC and IR constraints as well as gives the principal a strictly higher payoff than h^*, which is a contradiction.□

Step 2. For every $n \leq n^*$, the principal with type θ_n and type θ_n agent are indifferent between $h^*(\theta_{n^*})$ and $h^*(\theta_{n^*+1})$:

$$U_P(h^*(\theta_{n^*}), \theta_n) = U_P(h^*(\theta_{n^*+1}), \theta_n)$$
$$U_A(h^*(\theta_{n^*}), \theta_n) = U_A(h^*(\theta_{n^*+1}), \theta_n).$$

Proof. Due to the single crossing property $(\hat{S}_2)$, for every $n \leq n^*$ (24) and (26) imply

$$U_P(h^*(\theta_{n^*}), \theta_n) \leq U_P(h^*(\theta_{n^*+1}), \theta_n)$$
$$U_A(h^*(\theta_{n^*}), \theta_n) \leq U_A(h^*(\theta_{n^*+1}), \theta_n)$$

while (25) and (27) imply

$$U_P(h^*(\theta_{n^*}), \theta_n) \geq U_P(h^*(\theta_{n^*+1}), \theta_n)$$
$$U_A(h^*(\theta_{n^*}), \theta_n) \geq U_A(h^*(\theta_{n^*+1}), \theta_n).$$

Therefore, the above indifference relations are obtained.□

Hence, even after replacing $h^*(\theta_n)(= h^*(\theta_{n^*}))$ for every $n \leq n^*$ with $h^*(\theta_{n^*+1})$, the principal's payoff will be preserved without violating any IC constraints, which means that it is also an optimal IC mechanism under X. Therefore, this contradicts with the definition of n^*. ■

Then, a sufficient condition for the optimality of no screening under Z can be obtained by replacing the property of increasing conflict of interest in Theorem 1 with the condition $(\hat{S}_2)$.

Corollary 6 *Suppose that there exist a total order $\succeq_\Theta$ on Θ and an allocation subset $X \subseteq Z$ with a total order $\succeq_X$ s.t. with $(\succeq_X, \succeq_\Theta)$*

- *X is an incentive contract curve, and*
- *(Θ, X, U) is a single crossing problem with property $(\hat{S}_2)$.*

Then, for all type distributions $\mu \in \Delta\Theta$ there exists an optimal IC mechanism under Z, $h^* : \Theta \to X$ such that h^* is constant for θ.

E Proofs for the results in Section 5

E.1 Proof of Proposition 3

For the first part, it suffices to show that increasing MRS implies that for every $z' \succeq_{bd} z$ and $\theta' \succeq_\Theta \theta$,

$$U_A(z', \theta) \geq (>) U_A(z, \theta) \implies U_A(z', \theta') \geq (>) U_A(z, \theta') \tag{28}$$

$$U_A(z, \theta) \geq 0 \implies U_A(z, \theta') \geq 0 \tag{29}$$

$z' \succeq_{bd} z$ implies that $z'^+ \geq z^+$. Because it is trivial when $C^- = \emptyset$, I focus on the case when $C^- \neq \emptyset$.

(Case 1) $z^- > z'^-$

In this case, the first condition holds because both types of the agent have the same weak or strict preference for z over z'.

(Case 2) $z'^- \geq z^-$

First, increasing MRS implies the following:

$$\min_{1 \leq m \leq M} \frac{u_A^+(m, \theta')}{u_A^+(m, \theta)} \geq \max_{1 \leq n \leq N} \frac{u_A^-(n, \theta')}{u_A^-(n, \theta)}. \tag{30}$$

Also, note that $z'^+ \geq z^+$ and $z'^- \geq z^-$, *i.e*, for every m and n, $z'^+(m) \geq z^+(m)$ and $z'^-(n) \geq z^-(n)$. Hence,

$$U_A^+(z', \theta) - U_A^+(z, \theta) \geq 0 \text{ and } U_A^-(z', \theta) - U_A^-(z, \theta) \geq 0. \tag{31}$$

$$
\begin{aligned}
& U_A(z',\theta') - U_A(z,\theta') \\
=& (U_A^+(z',\theta') - U_A^+(z,\theta')) - (U_A^-(z',\theta') - U_A^-(z,\theta')) \\
\geq & \min_{1\leq m\leq M} \frac{u_A^+(m,\theta')}{u_A^+(m,\theta)} \cdot (U_A^+(z',\theta) - U_A^+(z,\theta)) - \max_{1\leq n\leq N} \frac{u_A^-(n,\theta')}{u_A^-(n,\theta)} \cdot (U_A^-(z',\theta) - U_A^-(z,\theta)) \\
\geq & \max_{1\leq n\leq N} \frac{u_A^-(n,\theta')}{u_A^-(n,\theta)} \cdot [(U_A^+(z',\theta) - U_A^+(z,\theta)) - (U_A^-(z',\theta) - U_A^-(z,\theta))] \\
= & \max_{1\leq n\leq N} \frac{u_A^-(n,\theta')}{u_A^-(n,\theta)} \cdot (U_A(z',\theta) - U_A(z,\theta))
\end{aligned}
$$

The first inequality holds because z' is component-wisely greater than z. The second one comes from (30) and (31). Therefore, we can obtain (28). Due to the same logic, (29) can be proved.

The second part comes directly from the single crossing property. More specifically, allocations given to two types of the agent by an IC mechanism have to be ordered by the standard componentwise order because if not, both types have the same strict preference over the allocations, and thus, IC constraints cannot hold.

E.2 Proof of Proposition 4

1. $C^- = \emptyset$; and for every m, $u_p^+(m,\theta)$ is nonincreasing in type; and $u_p^+(m,\theta) \leq 0$ whenever $u_A^+(m,\theta) = 0$.

   First note that when $C^- = \emptyset$, all the allocations are totally ordered with the standard component wise order $\geq$, which is also equivalent to $\succeq_{bd}$. Now, consider the following two cases and show that the increasing conflict of interest holds:

   - $x' \geq x$

     In this case, $U_P(x',\theta') \geq U_P(x,\theta')$, and thus, this implies that $U_P(x',\theta) \geq U_P(x,\theta)$ because

     $$U_P(x',\theta') - U_P(x,\theta') \geq U_P(x',\theta) - U_P(x,\theta).$$

     due to type-increasing $u_P^+(m,\theta)$. Thus, the increasing conflict of interest holds for such x and x'.

   - $x > x'$

     In this case, $U_A(x',\theta) \geq U_A(x,\theta)$ only if for every m such that $x(m) - x'(m) < 0$, $u_A^+(m,\theta) = 0$, which also implies $u_P^+(m,\theta) \leq 0$. Hence, $U_P(x,\theta) - U_P(x',\theta) \leq 0 = U_A(x,\theta) - U_A(x',\theta)$, and thus, the increasing conflict of interest property holds.

2. For every m, n, and θ, $u_P^+(m,\theta) > 0 > u_P^-(n,\theta)$ and $\frac{u_P^+(m,\theta)}{u_P^-(n,\theta)}$ is nondecreasing in type.

   - $x^+ \geq x'^+$ and $x'^- > x^-$
     $U_A(x',\theta) < U_A(x,\theta)$, and thus, the assumption of increasing conflict of interest does not hold.
   - $x^+ \geq x'^+$ and $x^- \geq x'^-$
     $U_P(x',\theta') < U_P(x,\theta')$ unless $x = x'$, and thus, again the assumption of increasing conflict of interest does not hold.
   - $(x'^+, x'^-) \geq (x^+, x^-)$,
     $U_P(x',\theta) \geq U_P(x,\theta)$ always holds because $u_P^+(m,\theta) > 0 > u_P^-(n,\theta)$ for every m and n. Thus, increasing conflict of interest holds.
   - $x'^+ > x^+$ and $x^- > x'^-$
     Because $\frac{u_P^+(m,\theta)}{u_P^-(n,\theta)}$ is nondecreasing in type, the principal's preference has the single crossing property, and thus, $U_P(x',\theta') \geq U_P(x,\theta')$ implies $U_P(x',\theta) \geq U_P(x,\theta)$. Hence, increasing conflict of interest holds.

E.3 Proof of Lemma 1

Well-definedness of π_{bd}

Because of the continuity of U_A, it suffices to show that

$$\left(U_A^+(\bar{z},\theta), U_A^-(\bar{z},\theta)\right) \geq \left(U_A^+(z,\theta), U_A^-(z,\theta)\right) \quad \forall (z,\theta) \in Z \times \Theta.$$

Also, because of the symmetry in the structures for positive and negative components, I show only that

$$U_A^+(\bar{z},\theta) \geq U_A^+(z,\theta) \quad \forall (z,\theta) \in Z \times \Theta.$$

For Z with the 0th dominance order, it is evident because $\bar{z}$ is component-wisely greater than z and each type agent's marginal benefit from each positive component is nonnegative.

For the next two cases of Z with the kth dominance order for $k = 1, 2$, the result directly comes from the same logic of a well-known fact about the stochastic dominance in footnote 15 because $U_A^+(z,\theta) = \sum_{m=1}^{M} z^+(m) u_A^+(m,\theta)$

E.4 Proof of Proposition 5

I define the payoff vectors of the principal and the agent which consist of their payoffs from each component of allocation. For each allocation z and type θ, the agent's *payoff vectors* respectively from the positive and negative components, $u_A^+(z,\theta) : C^+ \to \mathbb{R}_+$ and $u_A^-(z,\theta) : C^- \to \mathbb{R}_+$, are defined as $u_A^+(z,\theta)(m) \equiv z^+(m)u_A^+(m,\theta)$ and $u_A^-(z,\theta)(n) \equiv z^-(n)u_A^-(n,\theta)$ for m and n. Similarly, the principal's *payoff vectors* for z and θ, $u_P^+(z,\theta) : C^+ \to \mathbb{R}_+$ and $u_P^-(z,\theta) : C^- \to \mathbb{R}_+$, are also defined as $u_P^+(z,\theta)(m) \equiv z^+(m)u_P^+(m,\theta)$ and $u_P^-(z,\theta)(n) \equiv z^-(n)u_P^-(n,\theta)$ for m and n.

Stochastic dominance between payoff vectors

I first show that under the same conditions in Lemma 1, π_{bd} moves each allocation in a direction of decreasing its payoff vector in a stochastic dominance sense. Because of the symmetry in the structures for positive and negative components, I show only the stochastic dominance relation for the positive components. Let's first see

$$u_A^+(\pi_{bd}(z,\theta),\theta) \succeq_1 u_A^+(z,\theta),\ u_A^-(\pi_{bd}(z,\theta),\theta) \succeq_1 u_A^-(z,\theta)$$

in the following two cases:

- Z with the **0th** dominance order
- Z with the **1st** dominance order and (M_1)

Because the mapping $\pi_{bd}(\cdot,\theta)$ preserves type θ's total payoff from the positive components, it suffices to show that for every m

$$\sum_{m' \leq m} u_A^+(\pi_{bd}(z,\theta),\theta)(m) \geq \sum_{m' \leq m} u_A^+(z,\theta)(m) \tag{32}$$

where $u_A^+(z,\theta)(m) \equiv z^+(m)u_A^+(m,\theta)$.

First, if $m \geq \pi_{bd}(z,\theta)_0^+$, (32) holds because

$$\sum_{m' \leq m} u_A^+(\pi_{bd}(z,\theta),\theta)(m) = U_A^+(z,\theta) \geq \sum_{m' \leq m} u_A^+(z,\theta)(m). \tag{33}$$

When $m < \pi_{bd}(z,\theta)_0^+$, note that for every $m' \leq m$, $\pi_{bd}(z,\theta)^+(m') = \bar{z}^+(m')$. Thus, when Z is defined with the 0th dominance order,

$$\pi_{bd}(z,\theta)^+(m') \geq \bar{z}^+(m'),$$

and when Z is defined with the 1st dominance order,

$$\sum_{m' \leq m} \pi_{bd}(z,\theta)^+(m') \geq \sum_{m' \leq m} \bar{z}^+(m').$$

Hence, for Z with the 0th dominance order, (32) holds because $u_A^+(m,\theta) \geq 0$ for every m.

Also, for Z with the 1st dominance order, (32) holds because $u_A^+(m,\theta)$ is nonincreasing in $m \leq \pi_{bd}(z,\theta)_0^+$.

Next, in the case when Z is defined with the **2nd** dominance order and (M_2) holds, let's show that

$$u_A^+(\pi_{bd}(z,\theta),\theta) \succeq_2 u_A^+(z,\theta),\ u_A^-(\pi_{bd}(z,\theta),\theta) \succeq_2 u_A^-(z,\theta).$$

In this case, it suffices to show that for every m

$$\sum_{m' \leq m} (m'-m) u_A^+(\pi_{bd}(z,\theta),\theta)(m) \geq \sum_{m' \leq m} (m'-m) u_A^+(z,\theta)(m). \tag{34}$$

Again, because of (33), I only need to show this for $m < \pi_{bd}(z,\theta)_0^+$, which implies for every $m' \leq m$, $\pi_{bd}(z,\theta)^+(m') = \bar{z}^+(m')$. Thus, for $m < \pi_{bd}(z,\theta)_0^+$ and Z with the 2nd dominance order,

$$\sum_{m' \leq m} (m'-m)\pi_{bd}(z,\theta)^+(m') \geq \sum_{m' \leq m} (m'-m)\bar{z}^+(m').$$

By using the same logic of footnote 15, this implies that for a nonnegative valued, nonincreasing and convex function $g : \{1, \cdots, M\} \to \mathbb{R}_+$ with $g(m') = 0$ for $m' \geq \pi_{bd}(z,\theta)^+$,

$$\sum_{m' \leq m} g(m')\pi_{bd}(z,\theta)^+(m') \geq \sum_{m' \leq m} g(m')\bar{z}^+(m').$$

Moreover, $u_A^+(m,\theta)$ is nonincreasing and convex in $m \leq \pi_{bd}(z,\theta)_0^+$, and thus, this implies that $g(m') := (m'-m)u_A^+(m',\theta)|_{m' \geq m}$ is nonincreasing and convex in m' with $g(m') = 0$ for $m' \geq \pi_{bd}(z,\theta)_0^+$. Therefore, (34) holds for $m < \pi_{bd}(z,\theta)_0^+$.

Proof of Proposition 5

$X \subseteq Z$ is an incentive contract curve with a total order $\succeq_\Theta$ on Θ and $\pi_{bd} : Z \times \Theta \to X$ if for all $z \in Z$ and $\theta, \theta' \in \Theta$ with $\theta' \succeq_\Theta \theta$

$$\text{i} \begin{cases} U_A^+(\pi_{bd}(z,\theta),\theta') \leq U_A^+(z,\theta') \\ U_A^-(\pi_{bd}(z,\theta),\theta') \geq U_A^-(z,\theta') \end{cases}$$

ii $\begin{cases} U_P^+(\pi_{bd}(z,\theta),\theta) \geq U_P^+(z,\theta) \\ U_P^-(\pi_{bd}(z,\theta),\theta) \leq U_P^-(z,\theta) \end{cases}$.

Each of these implies $U_A(\pi_{bd}(z,\theta),\theta') \leq U_A(z,\theta')$ and $U_P(\pi_{bd}(z,\theta),\theta) \geq U_P(z,\theta)$, respectively.

Due to the same logic of footnote 15, because $U_A^+(z,\theta) = u_A^+(z,\theta') \cdot r_{\theta,\theta'}^+$, if $r_{\theta,\theta'}^+(m)$ is quasi-decreasing in m with respect to $\bar{z}_0^+$,

$$u_A^+(\pi_{bd}(z,\theta'),\theta') \succeq_1 u_A^+(z,\theta') \implies U_A^+(\pi_{bd}(z,\theta'),\theta) \geq U_A^+(z,\theta) = U_A^+(\pi_{bd}(z,\theta),\theta).$$

Hence, $\pi_{bd}(z,\theta')^+ \geq \pi_{bd}(z,\theta)^+$, and thus,

$$U_A^+(z,\theta') = U_A^+(\pi_{bd}(z,\theta'),\theta') \geq U_A^+(\pi_{bd}(z,\theta),\theta').$$

Similarly, if $r_{\theta,\theta'}^+(m)$ is quasi-decreasing in m with respect to $\bar{z}_0^+$ and convex in m for $m \leq \bar{z}_0^+$,

$$u_A^+(\pi_{bd}(z,\theta'),\theta') \succeq_2 u_A^+(z,\theta') \implies U_A^+(\pi_{bd}(z,\theta'),\theta) \geq U_A^+(z,\theta) = U_A^+(\pi_{bd}(z,\theta),\theta).$$

Hence, $\pi_{bd}(z,\theta')^+ \geq \pi_{bd}(z,\theta)^+$, and thus, again

$$U_A^+(z,\theta') = U_A^+(\pi_{bd}(z,\theta'),\theta') \geq U_A^+(\pi_{bd}(z,\theta),\theta').$$

Similarly, because $U_P^+(z,\theta) = u_A^+(z,\theta) \cdot r_{P,\theta}^+$,

$$u_A^+(\pi_{bd}(z,\theta),\theta) \succeq_1 u_A^+(z,\theta) \implies U_P^+(\pi_{bd}(z,\theta),\theta') \geq U_P^+(z,\theta')$$

if $r_{P,\theta}^+(m)$ is quasi-decreasing in m with respect to $\bar{z}_0^+$, and

$$u_A^+(\pi_{bd}(z,\theta),\theta) \succeq_2 u_A^+(z,\theta) \implies U_P^+(\pi_{bd}(z,\theta),\theta') \leq U_P^+(z,\theta')$$

if $r_{P,\theta}^+(m)$ is quasi-decreasing in m with respect to $\bar{z}_0^+$ and convex in m for $m \leq \bar{z}_0^+$

Likewise, the results for the marginal losses can be derived.

F Proofs for the results in Section 6

F.1 Proof of Corollary 1

It suffices to find a sufficient condition for the reducibility of (Θ, Z, U) to single crossing problem under X when Z is defined with the first dominance order. From Theorem 2, the following conditions need to be found. Fix a total order $\succeq_\Theta$ on Θ.

- Increasing MRS

  From Proposition 3, this holds if the valuation for the grand bundle $v_\theta(b^*)$ is nondecreasing in θ according to the order $\succeq_\Theta$: For $\theta' \succeq_\Theta \theta$,

$$v_{\theta'}(b^*) \geq v_\theta(b^*)$$

- Increasing conflict of interest property

  This holds because the principal's preference is type independent.

- Quasi-decreasing $u_A^+(m,\theta)$ and $u_A^-(n,\theta)$ in m and n with respect to $\bar{z}_0^+$ and $\bar{z}_0^-$ (Property (M_1))

  This is equivalent to that $u_A^+(m,\theta)$ is weakly nonincreasing with respect to the cutoff 1, *i.e.*, for every $\theta \in \Theta$ and $b \in B$,

$$v_\theta(b^*) \geq v_\theta(b).$$

- Quasi-decreasing $r_{\theta,\theta'}^+(m)$, $-r_{\theta,\theta'}^-(n)$, $r_{P,\theta}^+(m)$ and $-r_{P,\theta}^-(n)$ in m and n with respect to $\bar{z}_0^+$ and $\bar{z}_0^-$ for every $\theta' \succeq_\Theta \theta$ (Property (R_1))

  This is equivalent to that for every $\theta' \succeq_\Theta \theta$, $r_{\theta,\theta'}^+(m)$ is quasi-decreasing in m with respect to the cutoff 1: for every $b \in B$,

$$\frac{v_\theta(b^*)}{v_{\theta'}(b^*)} \geq \frac{v_\theta(b)}{v_{\theta'}(b)},$$

  which is equivalent to $\frac{v_{\theta'}(b^*)}{v_\theta(b^*)} \leq \frac{v_{\theta'}(b)}{v_\theta(b)}$.

F.2 Proof of Corollary 2

It suffices to show when (Θ, Z, U) is reducible to a single crossing problem under $X := \widehat{bd}(Z)$ when Z is defined with the 0th dominance order.. From Theorem 2, fixing a total order $\succeq_\Theta$ on Θ and the standard order $\geq$ on the set of goods $\{1, \cdots, d\}$, the following conditions need to be found:

- Increasing MRS

  For Proposition 3, this holds if for every $1 \leq m \leq d$ and θ, θ' with $\theta' \succeq_\Theta \theta$,

$$v_{\theta'}(m) \geq v_\theta(m).$$

- Increasing conflict of interest property

  This holds because the seller's payoff is type-independent.

- Quasi-decreasing $r^+_{\theta,\theta'}(m)$, $-r^-_{\theta,\theta'}(n)$, $r^+_{P,\theta}(m)$ and $-r^-_{P,\theta}(n)$ in m and n with respect to $\bar{z}_0^+$ and $\bar{z}_0^-$ for every $\theta' \succeq_\Theta \theta$ (Property (R_1))

  This is equivalent to that $r^+_{\theta,\theta'}(m) = \frac{v_\theta(m)}{v_{\theta'}(m)}$ is nonincreasing in m: for $m' \geq m$ and $\theta' \succeq_\Theta \theta$,

$$\frac{v_\theta(m)}{v_{\theta'}(m)} \geq \frac{v_\theta(m')}{v_{\theta'}(m')},$$

  which is equivalent to $\frac{v_{\theta'}(m')}{v_{\theta'}(m)} \geq \frac{v_\theta(m')}{v_\theta(m)}$.

It is evident that the same result holds for every permutation on $\{1, \cdots, d\}$.

F.3 Proof of Proposition 6

If part is shown in Corollary 2. For only if part, suppose not. That is, MRS between a pair of goods is not monotone according to $\succeq_\Theta$. For the ease of exposition, we denote as θ_n the nth lowest type according to the total order $\succeq_\Theta$. That is, $\Theta = \{\theta_1, \theta_2, \cdots, \theta_N\}$ for some $N > 0$ and $\theta_N \succeq_\Theta \cdots \succeq_\Theta \theta_2 \succeq_\Theta \theta_1$. According to the assumption of strictly increasing valuations, $v_{\theta_n}(k)$ is strictly increasing in n for every $k = 1, \cdots, d$. For each good $k = 1, \cdots, d$, denote as r_k a vector of the ratios beween the valuations of good k of θ_n and θ_{n+1} for every $n = 1, \cdots, N-1$:

$$r_k = (\frac{v_{\theta_2}(k)}{v_{\theta_1}(k)}, \frac{v_{\theta_3}(k)}{v_{\theta_2}(k)}, \cdots, \frac{v_{\theta_N}(k)}{v_{\theta_{N-1}}(k)})$$

One can define the lexicographic order on $\{r_1, \cdots, r_d\}$, which can be a partial order.

Also, For $1 \leq n \leq n' \leq N$, denote as $r^{n',n}$ a vector of the ratios beween the valuations of good k of $\theta_{n'}$ and θ_n for every $k = 1, \cdots, d$,:

$$r^{n',n} = (\frac{v_{\theta_{n'}}(1)}{v_{\theta_n}(1)}, \frac{v_{\theta_{n'}}(2)}{v_{\theta_n}(2)}, \cdots, \frac{v_{\theta_{n'}}(d)}{v_{\theta_n}(d)})$$

I define that for $n \leq n' \leq n''$ $r^{n',n}$ and $r^{n'',n'}$ are well-ordered if for every $1 \leq k \leq k' \leq d$, $(r^{n',n}(k), r^{n'',n'}(k))$ and $(r^{n',n}(k'), r^{n'',n'}(k'))$ are comparable to the lexicographic order $\geq_L$ on $\mathbb{R}^2_+$.

Then, the proof proceeds in the following steps.

Step 1 There exist three distinct states θ_n, $\theta_{n'}$ and $\theta_{n''}$ for $n \leq n' \leq n''$ such that $r^{n',n}$ and $r^{n'',n'}$ are not well-ordered.

Proof. First, that MRS between a pair of goods (k_1, k_2) is not monotone according to $\succeq_\Theta$ means that the lexicographic order over $\{r_1, \cdots, r_d\}$ is not a total order but a partial one, and thus, r_{k_1} and r_{k_2} are not comparable. This implies that there exists $1 < \bar{n} \leq N$ such that for $n < \bar{n}$, $r_{k_1}(n) \geq r_{k_2}(n)$ with a strict inequality for some n and $r_{k_1}(\bar{n}) < r_{k_2}(\bar{n})$. Then, this implies that $r^{\bar{n}-1,1}$ and $r^{\bar{n},\bar{n}-1}$ are not well-ordered because

$$r^{\bar{n},1}(k_1) > r^{\bar{n},1}(k_2),\ r^{\bar{n}+1,\bar{n}}(k_1) < r^{\bar{n}+1,\bar{n}}(k_2).$$

□

For the ease of exposition, I call the three states in Step 1, $\theta_3 \succeq_\Theta \theta_2 \succeq_\Theta \theta_1$. Then, I construct a three point distribution μ over the three states under which every optimal under which IC mechanism is not an upgrade pricing.

First, I relabel the indexes of good so that

1. $r^{3,2}(1) \leq r^{3,2}(2) \leq \cdots \leq r^{3,2}(d)$, and
2. if $r^{3,2}(k) = r^{3,2}(k+1)$ for some k, $r^{2,1}(k) \leq r^{2,1}(k+1)$.

Then, I define the following set D of the indexes:

$$D := \{k | \{k' | r^{2,1}(k') \leq r^{2,1}(k)\} \subsetneq \{k' | r^{3,2}(k') \leq r^{3,2}(k)\}\}.$$

Step 2 $D \neq \emptyset$

Proof. First, consider the largest index k satisfying that there exists k' s.t.

$$r^{2,1}(k) < r^{2,1}(k') \text{ and } r^{3,2}(k) > r^{3,2}(k'). \tag{35}$$

Due to Step 1, there exists an index k that satisfies (35). Also, for the largest k,

$$\{k' | r^{2,1}(k') \leq r^{2,1}(k)\} \neq \{k' | r^{3,2}(k') \leq r^{3,2}(k)\}.$$

Hence, if the largest k is not an element of D, this means that there exists an k'' s.t.

$$r^{2,1}(k'') \leq r^{2,1}(k) \text{ but } r^{3,2}(k'') > r^{3,2}(k).$$

Then, combined with (35), it implies that

$$r^{2,1}(k'') < r^{2,1}(k') \text{ and } r^{3,2}(k'') > r^{3,2}(k'),$$

which contradicts with the assumption that k is the largest index satisfying (35) for some k'. That is, such k is an element of D. □

Letting $\hat{r}^{3,2} := \max_{k \in D} r^{3,2}(k)$, the set of the indexes that has the value of $r^{3,2}$ equal to $\hat{r}^{3,2}$ can be written as

$$\{k^*, k^*+1, \cdots, k^* + l^{3,2}\} = \{k | r^{3,2}(k) = \hat{r}^{3,2}(k)\}$$

for some $k^* = \{1, \cdots, d\}$ and $l^{3,2} \geq 0$. According to the definition of D, this also implies that for every $k \leq k^* + l^{3,2}$,

$$r^{2,1}(k) \leq r^{2,1}(k^* + l^{3,2} + 1) \leq \cdots \leq r^{2,1}(n) \tag{36}$$

and there exist $l^{2,1} \leq l^{3,2}$ and a permutation σ on $\{1, \cdots, k^* - 1\}$ with some $k_1^* \leq k^* - 1$ s.t. for every $k \leq k_1^* < k'$,

$$r^{2,1}(k^*) = \cdots = r^{2,1}(k^* + l^{2,1}) < r^{2,1}(k^* + l^{2,1} + 1) \cdots \leq r^{2,1}(k^* + l^{3,2}) \tag{37}$$

$$r^{2,1}(\sigma(1)) \leq \cdots \leq r^{2,1}(\sigma(k_1^*)) \leq r^{2,1}(k^*) < r^{2,1}(\sigma(k_1^* + 1)) \leq \cdots \leq r^{2,1}(\sigma(k^* - 1)) \tag{38}$$

Now, I consider the following distribution $\mu = (\mu_1, \mu_2, \mu_3)$ over $\{\theta_1, \theta_2, \theta_3\}$ to show that upgrade pricing is not optimal for μ:

$$\mu_1 v_1(k^*) - \mu_2(v_2(k^*) - v_1(k^*)) > 0 > \mu_1 v_1(\sigma(k_1^*+1)) - \mu_2(v_2(\sigma(k_1^*+1)) - v_1(\sigma(k_1^*+1))) \tag{39}$$

$$\mu_2 v_2(k^*-1) - \mu_3(v_3(k^*-1) - v_2(k^*-1)) > 0 > \mu_2 v_2(k^*) - \mu_3(v_3(k^*) - v_2(k^*)). \tag{40}$$

Hence, if only the IR constraints and the local downward IC constraints are required, the optimal mechanism $\tilde{h}^*$ will assign type θ_1 a bundle $B_1 = \{\sigma(1), \cdots, \sigma(k_1^*), k^*, \cdots, k^* + l^{2,1}\}$, type θ_2 a bundle $B_2 = \{1, \cdots, k^* - 1\}$ and type θ_3 the grand bundle $B_3 = b^*$ with the prices that make all the local downward ICs and the IR for type θ_1 binding.

To show that upgrade pricing is not optimal for the type distribution μ, I consider the following two cases:

(Case 1) The valuation for each good is sufficiently convex in type:

$$\sum_{k \in B_2} v_3(k) - \sum_{k \in B_1} v_3(k) \geq \sum_{k \in B_2} v_2(k) - \sum_{k \in B_1} v_2(k) \tag{41}$$

(Case 2) The valuation for each good is not sufficiently convex in type:

$$\sum_{k\in B_2} v_3(k) - \sum_{k\in B_1} v_3(k) < \sum_{k\in B_2} v_2(k) - \sum_{k\in B_1} v_2(k) \tag{42}$$

In the following steps, I show that optimal LIC and IC mechanisms are equivalent in each of the above two cases. Moreover, in the first case, the optimal IC mechanism has only the local downward IC constraints binding but it is not an upgrade pricing. In the second case, it has all the downward IC constraints binding, which also means that upgrade pricing is not optimal due to the strictly monotone valuation or the resulting strict single crossing property.

(Step 3) In *Case 1*, $\tilde{h}^*$ satisfies all the constraints. That is, it is the optimal IC mechanism, which means that upgrade pricing is not optimal.

Proof. First, note that because of the strictly type-increasing valuations, the IR constraint for θ_1 and the IC constraints imply the IR constraints for the other types. Hence, it suffices to show that all the IC constraints are satisfied under $\tilde{h}^*$.

let's first see the local upward IC constraints. The upward IC between θ_2 and θ_3 holds because the downward IC between them is binding and the strictly monotone valuation implies that

$$\sum_{k\in b^*} v_3(k) - \sum_{k\in B_2} v_3(k) \geq \sum_{k\in b^*} v_2(k) - \sum_{k\in B_2} v_2(k).$$

The upward IC between θ_1 and θ_2 holds because the downward IC between them is binding and θ_2 has a higher relative valuation for goods in B_2/B_1 compared to goods in B_1/B_2 than θ_1.

Then, the remaining IC constraints are those between θ_1 and θ_3. First, because of (41), which implies that θ_3 has a stronger preference for B_2 over B_1 then θ_2. Also, because θ_2 has a stronger preference for $B_3 = b^*$ over B_2 than θ_1 due to the type-monotone valuations. This implies that the agent's preference over the three bundles, b^*, B_1, and B_2, has the single crossing property, and thus, the local IC constraints imply the non-local IC constraints.

Hence, all the IC constraints are satisfied under $\tilde{h}^*$, which means that $\tilde{h}^*$ is the optimal IC mechanism. Moreover, due to the definition of k^*, $B_1 \not\subseteq B_2$, and thus, upgrade pricing is not optimal.□

(Step 4) In *Case 2*, the optimal LIC mechanisms are the optimal IC mechanisms, and they have all the downward IC constraints binding. Hence, because of the strictly type-monotone valuations, the optimal IC mechanisms are not upgrade pricing.

Proof. Consider an optimal LIC mechanism h^*. It is evident that (1) h^* assigns θ_3 the grand bundle, and (2) the IR constraint for θ_1 and the downward IC constraint between θ_2 and θ_1 are binding. Moreover, at least one of the other two downward IC constraints, the constraint for θ_1 and θ_3 and the constraint for θ_2 and θ_3, is binding. In the former case, this contradicts with the optimality of h^* because the seller can extract more rent by selling more goods to θ_2. In the latter case, suppose that the downward IC between θ_3 and θ_1 is not binding. Then, h^* has to assign θ_2 the bundle B_2 and θ_1 a bundle larger than B_1 because if not, the seller can extract more rents by changing bundles assigned to θ_2 and θ_1 without violating any downward IC constraints. Then, (42) implies that downward IC between θ_3 and θ_1 does not hold, which contradicts with the assumption that h^* is an LIC mechanism. Therefore, all the downward ICs have to be binding under h^*.

It remains to show that h^* satisfies all the constraints. Again, it suffices to show that the upward IC constraint between θ_2 and θ_1 holds. Again, h^* has to assign θ_2 a bundle larger than B_2 and θ_1 a bundle smaller than B_1 because if not, the seller can extract more rents by changing bundles assigned to θ_2 and θ_1 without violating any downward IC constraints. The latter also implies that h^* has to assign θ_2 a bundle smaller than $\{1, \cdots, k^* + l^{2,1}\}(\supseteq B_1)$, because otherwise the strictly type monotone valuation implies that the downward IC between θ_3 and θ_1 is not binding. Due to the same logic, h^* has to assign θ_1 a bundle larger than $\{\sigma(1), \cdots, \sigma(k_1^*)\}(\subseteq B_2)$. That is, denoting as B_1^* and B_2^* the bundles assigned to θ_1 and θ_2 by h^*,

$$\begin{aligned}
\{\sigma(1), \cdots, \sigma(k_1^*)\} \subseteq & B_1^* \subseteq B_1 = \{\sigma(1), \cdots, \sigma(k_1^*), k^*, \cdots, k^* + l^{2,1}\} \\
\{1, \cdots, k^* - 1\} = B_2 \subseteq & B_2^* \subseteq \{1, \cdots, k^* + l^{2,1}\}
\end{aligned}$$

Hence, the upward IC between θ_1 and θ_2 holds because the downward IC between them is binding and θ_2 has a higher relative valuation for goods in B_2^*/B_1^* compared to goods in B_1^*/B_2^* than θ_1.□

F.4 Proof of Corollary 4

First, I show when (Θ, Z, U) is reducible to a single crossing problem under $X = \widehat{bd}(Z)$. According to Theorem 2, the lower contour set Z with the second order dominance order, the following conditions need to be obtained given a total order $\succeq_\Theta$ on Θ. According to Proposition 10, $(\hat{S}_2)$ instead of the increasing conflict of interest will be shown for the optimality of no screening. Here, I use the standard order $\geq$ as the total order.

- Increasing MRS

  This holds if $u_A(c, \theta) = (1 - G(c - \theta))(\mathbb{E}[\tilde{c}|\tilde{c} \geq c - \theta] - (c - \theta))$ is nondecreasing in θ, which is true because its first order derivative $\frac{\partial}{\partial \theta} u_A(c, \theta)$ is equal to $1 - G(c - \theta) \geq 0$.

- Quasi-decreasing and convex $u_A^+(m, \theta)$ and $u_A^-(n, \theta)$ in m and n with respect to $\bar{z}_0^+$ and $\bar{z}_0^-$. (Property (M_2))

  This again holds because $\frac{\partial}{\partial c} u_A(c, \theta) = -(1 - G(c - \theta)) \leq 0 \leq g(c - \theta) = \frac{\partial^2}{\partial c^2} u_A(c, \theta)$.

- Quasi-decreasing and convex $r_{\theta,\theta'}^+(c^+)$, $-r_{\theta,\theta'}^-(c^-)$, $r_{P,\theta}^+(c^+)$ and $-r_{P,\theta}^-(c^-)$ in c^+ and c^- with respect to $\bar{z}_0^+$ and $\bar{z}_0^-$ for every $\theta' \succeq_\Theta \theta$ (Property (R_2))

- Quasi-decreasing and convex $r_{\theta,\theta'}^+(c^+)$, $-r_{\theta,\theta'}^-(c^-)$, $r_{P,\theta}^+(c^+)$ and $-r_{P,\theta}^-(c^-)$ in c^+ and c^- with respect to $\bar{z}_0^+$ and $\bar{z}_0^-$ for every $\theta' \succeq_\Theta \theta$ (Property (R_2))

  As in footnote 17, I use a stronger version of Property (R_2) when $\bar{z}$ has the full support and $C^- = \emptyset$:

  $(\hat{R}_2)$ $\frac{u_P^+(c,\theta)}{u_A^+(c,\theta)}, \frac{\frac{\partial}{\partial c} u_P^+(c,\theta)}{\frac{\partial}{\partial c} u_A^+(c,\theta)}, , -\frac{\frac{\partial}{\partial c} u_A^+(c,\theta)}{u_A^+(c,\theta)}$, and $-\frac{\frac{\partial^2}{\partial c \partial \theta} u_A^+(c,\theta)}{\frac{\partial}{\partial c} u_A^+(c,\theta)}$ are nonincreasing in c.

  Because

  $$\frac{u_P^+(c,\theta)}{u_A^+(c,\theta)} = -\frac{\theta}{\mathbb{E}[\tilde{s}|\tilde{s} \geq c - \theta] - (c - \theta)}$$

  $$\frac{\frac{\partial}{\partial c} u_P^+(c,\theta)}{\frac{\partial}{\partial c} u_A^+(c,\theta)} = -\theta \frac{g}{1 - G}(c - \theta)$$

  $$-\frac{\frac{\partial}{\partial c} u_A^+(c,\theta)}{u_A^+(c,\theta)} = -\frac{1}{\mathbb{E}[\tilde{s}|\tilde{s} \geq c - \theta] - (c - \theta)}$$

  $$-\frac{\frac{\partial^2}{\partial c \partial \theta} u_A^+(c,\theta)}{\frac{\partial}{\partial c} u_A^+(c,\theta)} = -\frac{g}{1 - G}(c - \theta),$$

  all of these nonincreasingness hold if the hazard ratio $\frac{g}{1-G}(s)$ is nondecreasing in s. Especially, $\mathbb{E}[\tilde{s}|\tilde{s} \geq c - \theta] - (c - \theta)$ are also nonincreasing in c

- $(\hat{S}_2)$

  It suffices to find when the single crossing property of $U|_{X\times\Theta}$ with $(\geq, \succeq_\Theta^D)$ holds. This holds because $u_A(c,0) = u_P(c,0)$ for every c and

$$\frac{\partial}{\partial\theta}u_A(c,\theta) = 1 - G(c-\theta) \geq 0 \geq -\theta g(c-\theta) = \frac{\partial}{\partial\theta}u_P(c,\theta).$$

F.5 Proof of Corollary 3

I first show when (Θ, Z, U) is reducible to a single crossing problem under $X = \hat{bd}(Z)$ by using a stronger version of Theorem 2. Proposition 7 shows that increasing conflict of interest can be replaced with (10). Then, because Z is the lower contour set Z of $\mathbb{1}$ with respect to the 0th dominance order, only the following conditions need to be found for a given total order $\succeq_\Theta$ on Θ:

- Increasing MRS

  This holds if for every $s' > 0 > s$, $\frac{u_A^+(s',\theta)}{u_A^-(-s,\theta)} = -\frac{f(s',\theta)u(s')}{f(s,\theta)u(s)}$ is nondecreasing in θ according to the order $\succeq_\Theta$. Because $u(s') > 0 > u(s)$, it holds if $\frac{f(s',\theta)}{f(s,\theta)}$ is nondecreasing in θ, that is, $\frac{f(s,\theta')}{f(s,\theta)} \leq \frac{f(s',\theta')}{f(s',\theta)}$ for every $\theta' \succeq_\Theta \theta$.

- Quasi-decreasing $r^+_{\theta,\theta'}(c^+)$, $-r^-_{\theta,\theta'}(c^-)$, $r^+_{P,\theta}(c^+)$ and $-r^-_{P,\theta}(c^-)$ in c^+ and c^- with respect to $\bar{z}_0^+$ and $\bar{z}_0^-$ for every $\theta' \succeq_\Theta \theta$ (Property (R_1))

  For every $\theta' \succeq_\Theta \theta$,

  $r^+_{\theta,\theta'}(c^+) := \frac{u_A^+(c^+,\theta)}{u_A^+(c^+,\theta')} = \frac{f(c^+,\theta)}{f(c^+,\theta')}$: nonincreasing in c^+

  $r^-_{\theta,\theta'}(c^-) := \frac{u_A^-(c^-,\theta)}{u_A^-(c^-,\theta')} = \frac{f(-c^-,\theta)}{f(-c^-,\theta')}$: nondecreasing in c^-

  $r^+_{P,\theta}(c^+) := \frac{u_P^+(c^+,\theta')}{u_A^+(c^+,\theta)} = \frac{v(c^+)}{u(c^+)}$: nonincreasing in c^+

  $r^-_{P,\theta}(c^-) := \frac{u_P^-(c^-,\theta')}{u_A^-(c^-,\theta)} = \frac{v(-c^-)}{u(-c^-)}$: nondecreasing in c^-

  The first two conditions with the above condition for increasing MRS are equivalent to (8) while the last two are equivalent to (9).

Moreover, due to the single crossing condition of (Θ, X, U), every IC mechanism h under X has to satisfy that for $\theta' \succeq_\Theta \theta$,

$$h(\theta) \subseteq h(\theta'), \tag{43}$$

which means that h takes the form of the nested acceptance interval structure. Therefore, the optimal IC mechanism under X, which is also an optimal IC mechanism under Z, also satisfies the nested acceptance interval structure.

Lastly, I need to show the implementability of such IC mechanism h through an information structure. Because $U_A(h(\theta'), \theta') \geq U_A(h(\theta), \theta')$ and $U_A(h(\theta), \theta) \geq U_A(h(\theta'), \theta)$ for $\theta' \succeq_\Theta \theta$, the single crossing property implies with (43) that for every $\theta'' \succeq_\Theta \theta'$,

$$U_A(h(\theta'), \theta'') \geq U_A(h(\theta), \theta'')$$

and for every $\theta \succeq_\Theta \theta''$,

$$U_A(h(\theta), \theta'') \geq U_A(h(\theta'), \theta'')$$

Hence, consider an information structure under which S is partitioned into

$$\{h(\theta_n) \backslash h(\theta_{n-1}) | n = 1, \cdots, N+1\}$$

where $\Theta = \{\theta_N, \cdots, \theta_1\}$ with $\theta_N \succeq_\Theta \cdots, \succeq_\Theta \theta_1$, $h(\theta_0) \equiv \emptyset$, and $h(\theta_{N+1}) \equiv S$. Then, for every $n' \leq n \leq n''$

$$U_A(h(\theta_{n'}) \backslash h(\theta_{n'-1}), \theta_n) \geq U_A(\emptyset, \theta_n) \geq U_A(h(\theta_{n''}) \backslash h(\theta_{n''-1}), \theta_n)$$

Therefore, each agent with type θ_n will choose the acceptance set $h(\theta_n)$, which implies that every nested interval structure of acceptance sets from an IC mechanism h can be implemented by the information structure.

G Weakly increasing conflict of interest property

G.1 General result

In Section 4.4, I provided a sufficient condition for the equivalence between optimal LIC and IC mechanisms in single crossing screening problem. Basically, the condition says that for a set of pairs of allocations that can be used by an optimal LIC mechanism with violating an upward IC constraint, the principal is less likely to have the same preference with the agent as type goes up. In fact, this condition can be weakened for some of the pairs of allocations. First, I define the following allocation subset X_s.

Definition 12 *In a single crossing screening problem* (Θ, X, U)*,* X_s *is the set of allocations that satisfies the following:*

For every $x \in X_s$ *and every* $\epsilon > 0$*, there exists an* $x'' \in X$ *satisfying that* $|x - x''| < \epsilon$ *and for every* $\theta \in \Theta$

$$U_P(x'', \theta) > U_P(x, \theta) \text{ and } U_A(x, \theta) \geq U_A(x'', \theta).$$

In other words, X_s is the set of allocation x such that there exists a direction v with which moving x decreases the agent's payoff but increases the principal's payoff regardless of the agent's type. For example, in single good monopoly problem where the seller maximizes the expected revenue, an increase in price is the direction of the allocation that leads to the decrease in the buyer's payoff and the increase in the seller's payoff. Hence, $X_s = X$.

Such allocation can be assigned to a type of agent by an optimal LIC mechanism only when a downward IC constraint of the type of the agent is binding. If not, the principal's payoff can be strictly increased without violating any downward IC constraint. Hence, the property of increasing conflict of interest can be weakened for $x \in X_s$.

Definition 13 *A screening problem* (Θ, X, U) *has* ***weakly increasing conflict of interest*** *with a total order* $\succeq_\Theta$ *on* Θ *if the followings hold together:*

- *When* $x \in X_s$*, for every* $x' \in X$ *and* $\theta, \theta' \in \Theta$ *with* $\theta \succeq_\Theta \theta'$*,*

$$\begin{cases} U_A(x, \theta) = U_A(x', \theta) \\ U_A(x, \theta') \geq U_A(x', \theta') \\ U_P(x, \theta) \geq U_P(x', \theta) \end{cases} \implies U_P(x, \theta') \geq U_P(x', \theta').$$

- *When* $x \notin X_s$*, for every* $x' \in X$ *and* $\theta, \theta' \in \Theta$ *with* $\theta \succeq_\Theta \theta'$*,*

$$\begin{cases} U_A(x, \theta) \geq U_A(x', \theta) \\ U_A(x, \theta') \geq U_A(x', \theta') \\ U_P(x, \theta) \geq U_P(x', \theta) \end{cases} \implies U_P(x, \theta') \geq U_P(x', \theta').$$

Proposition 11 *Suppose that* (Θ, X, U) *is a single crossing problem and has weakly increasing conflict of interest with* $(\succeq_X, \succeq_\Theta)$*. Then, for all type distributions* $\mu \in \Delta\Theta$*, every optimal IC mechanism under* X *is an optimal LIC mechanism under* X *with* $\succeq_\Theta$*.*

Proof. The proof is same as the proof of Proposition 2 up to Step 3. Again, given an agent-optimal optimal LIC mechanism h under X, let n^* be the smallest n that does not satisfy the local upward IC constraint between θ_n and θ_{n+1}. The only difference is that the local downward IC constraint between between θ_{n^*} and θ_{n^*+1} has to be binding if $h(\theta_{n^*+1}) \in X_s$. If not, the principal's payoff is strictly increased by replacing $h(\theta_{n+1})$ with another allocation $x'' \in X$ s.t.

- x'' is more preferred to $h(\theta_{n+1})$ by the principal and less preferred by the agent regardless of the agent's type, and
- $|x'' - h(\theta_{n+1})|$ is sufficiently small so that the agent with type θ_{n+1} still prefers x'' to $h(\theta_n)$.

First, the existence of such x'' is guaranteed by $h(\theta_{n+1}) \in X_s$. Also, due to (21) in the proof of Proposition 2, the second condition also means that he prefers x'' to $h(\theta_m)$ for every $m \leq n$. Hence, the change in allocation increases the principal's payoff without violating any downward IC constraint, which contradicts with the optimality of h. Therefore, in this case when $h(\theta_{n+1}) \in X_s$, the local downward IC constraint is binding, and thus, (23) is replaced with the binding constraint:

$$U_A(h(\theta_{n^*+1}), \theta_{n^*+1}) = U_A(h(\theta_{n^*}), \theta_{n^*+1}).$$

Hence, the first condition in the weakly increasing conflict of interest property is enough to show a contradiction. ■

The weakly increasing conflict of interest property is equivalent to the increasing conflict of interest property if $X_s = \emptyset$. Moreover, the former is a weaker condition than the latter because the condition for $x \in X_s$ is weaker than the condition for $x \notin X_s$ in that one only needs to consider x' that gives the same payoff to type θ as x does. Especially, in a strict single crossing problem (Θ, X, U) with $(\succeq_\Theta, \succeq_X)$, this also implies $x' \succeq_X x$.[37]

Remark 4 *When money transfers are allowed, $X_s = X$. Moreover, the first condition of weakly increasing conflict of interest property can be even further weakened by replacing the principal's payoff with the social welfare in its last inequality because the principal can allocate*

[37] A screening problem is a strict single crossing problem if it is a single crossing problem and all the conditions in Definition 1 also hold for strict inequalities

the welfare to the players however she wants: For every $x, x' \in X$ and $\theta, \theta' \in \Theta$ with $\theta \succeq_\Theta \theta'$

$$\begin{cases} U_A(x,\theta) = U_A(x',\theta) \\ U_A(x,\theta') \geq U_A(x',\theta') \\ U_P(x,\theta) \geq U_P(x',\theta) \end{cases} \implies U_P(x,\theta') + U_A(x,\theta') \geq U_P(x',\theta') + U_A(x',\theta').$$ [38]

Proof. *As in Proposition 11, the proof is same up to Step 3 of Proposition 2, and hence, it suffices to check that for an agent-optimal optimal LIC mechanism h^* under X this property has to hold when $\theta = \theta_{n+1}$, $\theta' = \theta_n$, $x = h(\theta_{n+1})$ and $x' = h(\theta_n)$. If this does not hold, h^* is not agent-optimal: By replacing $h(\theta_n)$ with $h(\theta_{n+1}) - t$ for a transfer $t > 0$ with $U_A(h(\theta_n), \theta_n) = U_A(h(\theta_{n+1}) - t, \theta_n)$, without violating any downward IC constraint either the principal's payoff can be strictly increased or the variance among the allocations to the highest types can get smaller.* ■

Therefore, the main theorem still holds even if increasing conflict of interest property is replaced with its weak version.

Theorem 4 *Suppose that there exist a total order $\succeq_\Theta$ on Θ and an allocation subset $X \subseteq Z$ with a total order $\succeq_X$ s.t. with $(\succeq_X, \succeq_\Theta)$*

- *X is an incentive contract curve, and*
- *(Θ, X, U) is a single crossing one and has weakly increasing conflict of interest.*

Then, (Θ, Z, U) is reducible to a single crossing problem under X.

G.2 Weakly increasing conflict of interest in the setting of Section 5

Assuming the additional structure as in Section 5, one can obtain the following sufficient conditions for the weakly increasing conflict of interest property.

Proposition 12 *Suppose that type set Θ has increasing MRS between positive and negative components with $\succeq_\Theta$. Also, for some $0 \leq \bar{n} \leq N$, the principal's marginal benefits from all positive components are positive while her marginal loss from negative component $-n$ is negative for $n \leq \bar{n}$ and otherwise positive: For every θ, m and $n \leq \bar{n} < n'$,*

$$u_P^+(m,\theta) > 0, \; u_P^-(n,\theta) < 0 \leq u_P^-(n',\theta).$$

[38] This is equivalent to the single crossing property of the social welfare assumed in Yang (2021).

Then, for $X = \hat{bd}(Z)$, the weakly increasing conflict of interest property holds if for all m, $n > \bar{n}$ and $\theta \succeq_\Theta \theta'$,

$$\frac{u_A^+(m,\theta')}{u_A^-(n,\theta')} \le \frac{u_P^+(m,\theta')}{u_P^-(n,\theta')} \le \frac{u_P^+(m,\theta)}{u_P^-(n,\theta)}$$

Proof. Let $C_P^+ := \{1, \cdots, \bar{n}\} \subseteq C^-$. The principal has positive marginal payoff from every negative component in C_P^+. Then, it is straightforward to show that

$$X_s \supseteq \tilde{X}_s := \{x \in \hat{bd}(Z) | (\bar{z}^+, \bar{z}^-|_{C_P^+}) \succeq_{bd} x \text{ and } \bar{z}^-(n) > x^-(n) \text{ for some } 1 \le n \le \bar{n}\}$$

because for the allocation $x \in \tilde{X}_s$, the principal's payoff can be strictly increased with decreasing all types of the agent's payoffs by raising the value of $x^-(n)$. Now, consider the following two cases and show that the weakly increasing conflict of interest holds.

- $x \in \tilde{X}_s$

  First, $U_A(x,\theta) = U_A(x',\theta)$ implies that either $x \ge x'$ or $x' \ge x$ because otherwise the equality cannot hold. Also, if $x \ge x'$, then $U_P(x,\theta') \ge U_P(x',\theta')$ because $x^-(n) \ge 0$ only for $n \in C_P^+$ and for every positive component and every negative component in C_P^+ the principal have nonnegative marginal payoffs.

  It only remains to show that the first part of $(\hat{S})$ holds when $x' \ge x$. Because

  $$\frac{u_A^+(m,\theta)}{u_A^-(n,\theta)} \le \frac{u_P^+(m,\theta)}{u_P^-(n,\theta)},$$

  $U_A(x,\theta) = U_A(x',\theta)$ implies $U_P(x',\theta) \ge U_P(x,\theta)$ due to the same logic of single crossing property. Moreover, because $\bar{z}^-(n) > x^-(n)$ for some $1 \le n \le \bar{n}$, this inequality has to strictly hold, which does not satisfy the assumption of the first part of the weakly increasing conflict of interest property.

- $x \notin \tilde{X}_s$

  First, because $U_A(x,\theta) \ge U_A(x',\theta)$ in the second part of the weakly increasing conflict of interest property (that is, the increasing conflict of interest), it cannot be the case that $x^+ \le x'^+$ and $x^- \ge x'^-$. Also, in the opposite case where $x^+ \ge x'^+$ and $x^- \le x'^-$, $U_P(x,\theta') \ge U_P(x',\theta')$ always holds, which means that the increasing conflict of interest always holds. Hence, the remaining two cases are $(x^+, x^-) \ge (x'^+, x'^-)$ and $(x^+, x^-) \le (x'^+, x'^-)$.

– $(x^+, x^-) \geq (x'^+, x'^-)$
Because $\frac{u_A^+(m,\theta')}{u_A^-(n,\theta')} \leq \frac{u_P^+(m,\theta')}{u_P^-(n,\theta')}$, the single crossing property implies the following:

$$U_A(x, \theta') \geq U_A(x', \theta') \implies U_P(x, \theta') \geq U_P(x', \theta').$$

Thus, the increasing conflict of interest holds.

– $(x^+, x^-) \leq (x'^+, x'^-)$
Because $\frac{u_P^+(m,\theta')}{u_P^-(n,\theta')} \leq \frac{u_P^+(m,\theta)}{u_P^-(n,\theta)}$, again the single crossing property implies the following:

$$U_P(x, \theta) \geq U_P(x', \theta) \implies U_P(x, \theta') \geq U_P(x', \theta').$$

Thus, the increasing conflict of interest holds.

■

G.3 Proof of Proposition 7

Proposition 7 is a corollary of the continuous version of Proposition 12. Because for $s \leq s_0 \leq 0 \leq s'$ $\frac{u_A^+(s',\theta')}{u_A^-(-s,\theta')} = -\frac{f(s',\theta')}{f(s,\theta')}\frac{u(s')}{u(s)}$, $\frac{u_P^+(s',\theta')}{u_P^-(-s,\theta')} = -\frac{f(s',\theta')}{f(s,\theta')}\frac{v(s')}{v(s)}$,

$$\frac{u_A^+(s',\theta')}{u_A^-(-s,\theta')} \leq \frac{u_P^+(s',\theta')}{u_P^-(-s,\theta')} \leq \frac{u_P^+(s',\theta)}{u_P^-(-s,\theta)}$$

implies that $\frac{u(s')}{u(s)} \geq \frac{v(s')}{v(s)}$ and $\frac{f(s',\theta')}{f(s,\theta')} \geq \frac{f(s',\theta)}{f(s,\theta)}$. The latter comes from the single crossing condition of (Θ, X, U).

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