Ex-post approaches to prioritarianism and sufficientarianism*

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Abstract. Although sufficientarianism has been gaining interest as a theory of distributive justice in recent years, it has not been examined in the presence of risk. We propose an ex-post approach to sufficientarianism that has a strong link to ex-post prioritarianism. Both ex-post criteria are based on an axiom that we refer to as prospect independence of the unconcerned, a natural extension of the independence axiom known from the literature that focuses on situations with no risk. We characterize a class of ex-post prioritarian orderings as well as the corresponding class of ex-post sufficientarian orderings. In addition, we point out some important differences between these two ex-post criteria, and we examine how they fare when assessed in terms of specific ex-ante Paretian axioms.

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1 Introduction

An ethical theory labeled sufficientarianism has been analyzed philosophically as distinct from utilitarianism, prioritarianism, or egalitarianism (Frankfurt, 1987; Crisp, 2003; Brown, 2005; Casal, 2007; Hirose, 2016). Its distinctive feature is the use of a threshold that represents sufficiency. The threshold is a utility level such that an individual is deemed to have enough if and only if his or her utility reaches this level. Roughly speaking, the primary concern of this ethical theory is to minimize insufficiency among individuals. Axiomatic foundations of sufficientarian theories have recently been provided in contributions such as Alcantud, Mariotti, and Veneziani (2022) and Bossert, Cato, and Kamaga (2022, 2023).

Traditional sufficientarian approaches do not pay much (if any) attention to the presence of risk. However, most public-policy choices involve considerable risk as far as the outcomes that eventually materialize are concerned and, therefore, there appears to be a need to go beyond the riskless case. We propose to do so by utilizing a welfarist framework of social evaluation of state-contingent alternatives. This framework includes a fixed probability distribution over the set of states. Ex-post utility distributions that occur in each state are assumed to be variable, and a profile of all individuals at all states is called a prospect. We examine an ordering (that is, a reflexive, complete, and transitive binary relation) defined on the set of prospects. Fleurbaey (2010) provides a new ex-post welfare criterion relying on this framework, and related approaches can be found in Blackorby, Bossert, and Donaldson (2002, 2005). Originally introduced by Arrow (1964) in the context of individual decision-making, Blackorby, Davidson, and Donaldson (1977) establish a foundation of the expected-utility hypothesis.

The sufficientarian principle that we characterize in this paper is what we refer to as *ex-post sufficientarianism.* This principle emphasizes the depth of insufficiency from an ex-post perspective rather than focusing on the expected-utility level evaluated from an ex-ante perspective. Clearly, the expected insufficiency of ex-post utilities is significantly different from the insufficiency of individual expected utilities. If one considers the insufficiency of expected utilities, it does not matter if the utility is significantly below the threshold as long as the associated probability is very small. Under ex-post sufficientarianism, as long as some ex-post utilities fall below the threshold, it is always considered a significant problem.

Ex-post sufficientarianism is closely related to *ex-post prioritarianism,* which has been extensively studied in the context of social evaluation in risky situations. Prioritarianism is a method of social evaluation that gives inequality-averse weights to the individuals according to their levels of utility by using an increasing and strictly concave transformation; see Rabinowicz (2002), McCarthy (2008), Adler (2018, 2019), Adler and Holtug (2019), and Adler and Norheim (2022). According to ex-post prioritarianism, a prospect is better when the expected value of the weighted sum of ex-post utilities is higher. This thought is advocated by Adler and Sanchirico (2006); see also Adler (2012). To the best of our knowledge, there has been no axiomatic characterization of ex-post prioritarianism so far.

We employ a unified method to characterize ex-post prioritarianism and ex-post sufficientarianism. Our key axiom is what we call prospect independence of the unconcerned, the ex-post variant of well-established independence properties that are familiar from the literature on social evaluation without risk. Individuals who face the same risk in two prospects are called unconcerned, and the axiom requires that the social comparison of these two prospects is independent of unconcerned individuals.

Our first main result consists of a characterization of the class of ex-post prioritarian orderings by combining prospect independence of the unconcerned with strong Pareto for no risk, continuity, anonymity, the Pigou-Dalton transfer principle for no risk, and the social expected-utility hypothesis. Strong Pareto for no risk, continuity, anonymity, and the Pigou-Dalton transfer principle for no risk are standard axioms, which are commonly used when characterizing prioritarian orderings in a framework with no risk. The social expected-utility hypothesis requires the existence of a social von Neumann-Morgenstern function such that prospects are ranked in terms of the expected values of ex-post social welfare.

We then proceed to a characterization of ex-post sufficientarianism. In addition to prospect independence of the unconcerned, there is another key axiom for the characterization that is intended to capture the distinctive nature of sufficientarianism. Sufficientarian theories are primarily concerned with changes in utilities below the threshold but that does not mean that utilities above the threshold do not matter and, therefore, sufficientarian theories can very well be compatible with Paretian axioms. Sufficientarianism puts unequivocal priority on the utilities below the threshold and uses those above the threshold as a tie-breaking device. We formalize this attribute as an axiom that we label ex-post absolute priority. This axiom is a natural extension of the axiom of absolute priority proposed by Bossert, Cato, and Kamaga (2022, 2023), who present an axiomatic characterization of classes of sufficientarian social orderings.

Ex-post absolute priority is not compatible with the social expected-utility hypothesis, a fundamental property of ex-post prioritarianism. This incompatibility is caused by the lexical priority assigned to those below the threshold by the axiom of ex-post absolute priority. In other words, the existence of a sufficiency threshold does not allow us to apply the social expected-utility hypothesis across this threshold. However, if the social expected-utility hypothesis is restricted to utilities below the threshold and above the threshold separately, it is compatible with ex-post absolute priority.

We characterize the class of ex-post sufficientarian orderings by using prospect independence of the unconcerned, ex-post absolute priority, and the restricted social expected-utility hypothesis, in addition to strong Pareto for no risk, anonymity, and two restricted continuity axioms. It is well-known that most sufficientarian theories are not compatible with full continuity and, thus, only restricted versions such as continuity below the threshold and continuity above the threshold can be satisfied; see Roemer (2004) and Bossert, Cato, and Kamaga (2022, 2023) for detailed discussions.

To put our contribution into perspective, we note first that the issue of social evaluation with risk has been an important topic since the pioneering contribution of Harsanyi (1955) who provides a formal foundation of utilitarianism; see also Blackorby, Donaldson, and Weymark (1999). Diamond (1967) raises an ex-ante equality issue that applies to Harsanyi's arguments. Hammond (1983) and Broome (1991) provide early observations on ex-post criteria, which are substantially developed by Rabinowicz (2002), Adler and Sanchirico (2006), and Adler (2012) as ex-post prioritarianism.

Notably, both ex-post prioritarian and ex-post sufficientarian orderings violate the ex-ante Pareto principle, which requires that a prospect is better than another if each individual's expected utility in the former is higher than in the latter. Fleurbaey (2010) proposes a weakening of the ex-ante Pareto principle, weak Pareto for equal risk, according to which the ex-ante Pareto principle applies to prospects where all individuals face the same risk. Using this axiom, Fleurbaey (2010) provides a characterization of what is called the class of expected equally-distributed-equivalent (EDE) social orderings. We highlight some differences between expected EDE social orderings and ex-post social orderings. The axiomatic analysis of Fleurbaey (2010) is extended by Fleurbaey and Zuber (2013); see also Fleurbaey, Gajdos, and Zuber (2015) as well as Mongin and Pivato (2015). In particular, Fleurbaey and Zuber (2013) use an independence property similar to ours to offer a joint characterization of the utilitarian ordering and a specific multiplicative form.

Section 2 introduces the formal setting employed in this paper. Our basic axioms are defined and discussed in Section 3. Section 4 contains our results on ex-post prioritarian social evaluation, and Section 5 is devoted to ex-post sufficientarian criteria. Sections 6 and 7 examine the relationship with ex-ante Paretian requirements. Section 8 concludes. The independence of the axioms used in our main characterization results is established in the Appendix.

2 Setting

For $r \in \mathbb{N}$, we use $\mathbf{1}_r$ to denote the r-dimensional vector composed of r ones. We consider a welfarist framework of social evaluation of state-contingent alternatives. Let $S = \{1, \ldots, m\}$ be the finite set of $m \geq 2$ states and $(\pi^s)_{s \in S}$ be an exogenously given fixed probability distribution on states $s \in S$. We assume that $\pi^s > 0$ for all $s \in S$ and $\sum_{s \in S} \pi^s = 1$. This assumption involves no loss of generality as long as there are at least two states with positive probabilities because any state with a probability of zero may be dropped. The finite set of individuals is given by $N = \{1, \ldots, n\}$, where $n \geq 3$ is assumed.

Let u_i^s denote the ex-post utility level of individual i in state s. Social alternatives to be evaluated are given by prospects. A prospect is a profile of all individuals' utilities in all states and is denoted by $u = (u_i^s)_{i \in N, s \in S}$. The set of all prospects is $\mathcal{D} = \mathbb{R}^{mn}$. Given a prospect $u = (u_i^s)_{i \in N, s \in S}$, let $u^s = (u_1^s, \ldots, u_i^s, \ldots, u_n^s) \in \mathbb{R}^n$ be the prospect in state $s \in S$. Similarly, $u_i = (u_i^1, \ldots, u_i^s, \ldots, u_i^m) \in \mathbb{R}^m$ represents individual i's prospect. For all $u \in \mathcal{D}$ and for all $i \in N$, let $E(u_i)$ be the expected value $E(u_i) = \sum_{s \in S} \pi^s u_i^s$ of individual i's ex-post utilities.

A subdomain of $\mathcal{D}$ is considered in our analysis. A prospect u such that $u^s = u^{s'}$ for all $s, s' \in S$ does not include any risk. Such a prospect is called *riskless*. Let $\mathcal{D}^c$ be the set of riskless prospects. For all $u \in \mathcal{D}$ and for all $s \in S$, let $[u^s] = (u^s, \ldots, u^s) \in \mathcal{D}^c$ denote a riskless prospect such that u^s occurs in each state $s' \in S$. We note that, for each riskless prospect $u \in \mathcal{D}^c$, there exists a prospect $u^s \in \mathbb{R}^m$ in state s such that $[u^s] = u$. Furthermore, if $u \in \mathcal{D}^c$, then $E(u_i) = u_i^s$ for all $s \in S$ and for all $i \in N$.

The sufficiency threshold $\theta \in \mathbb{R}$ is an exogenously given threshold level of utility. A given threshold θ is common to all states $s \in S$ and applies to ex-post utility levels in all states. Its interpretation is that, for each state $s \in S$, those individuals whose ex-post utilities are on or above the threshold are deemed to have enough. For all $u \in \mathcal{D}$ and for all $s \in S$, we define the sets of those individuals whose utility is lower than and higher than the threshold θ in state s by

$$L(u^s) = \{i \in N \mid u_i^s < \theta\};$$

$$H(u^s) = \{i \in N \mid u_i^s > \theta\}.$$

A social ordering for prospects is a reflexive, complete, and transitive binary relation R on $\mathcal{D}$. For two prospects $u, v \in \mathcal{D}$, we write uRv instead of $(u, v) \in R$ to indicate that u is at least as good as v. The asymmetric and symmetric parts of R are denoted by P and I.

A social ordering R is *ex-post generalized utilitarian* if and only if there exists a continuous and increasing function $g\colon \mathbb{R} \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}$, uRv if and only if

$$\sum_{s \in S} \pi^s \sum_{i \in N} g(u_i^s) \geq \sum_{s \in S} \pi^s \sum_{i \in N} g(v_i^s).$$

An important subclass of these principles consists of the *ex-post prioritarian* criteria, which are obtained by choosing a strictly concave transformation g in the definition of the ex-post generalized utilitarian orderings. Ex-post prioritarianism is a very natural extension of prioritarianism to the evaluation of risky situations because a prioritarian evaluation applies to each state $s \in S$.

An alternative special case of ex-post generalized utilitarianism is *ex-post utilitarianism,* which is associated with a linear transformation g. Harsanyi (1955) characterizes ex-post utilitarianism as an ordering defined on the set of lotteries. Utilitarianism for prospects is characterized by Blackorby, Bossert, and Donaldson (2005) who employ an ex-ante approach to evaluating prospects.

Brown (2005), Hirose (2016), and Bossert, Cato, and Kamaga (2022, 2023) develop sufficientarian orderings in a framework that does not involve risk. Their sufficientarian principles are based on a lexicographic procedure. The primary criterion employed consists of the total gap between (transformed) utilities and the sufficiency threshold for those below the threshold. If these gaps are equal for two distributions, the corresponding gap for those above the threshold is consulted. These orderings are compatible with the Pareto principle. We extend their formulation of sufficientarianism to the evaluation of risky situations, which applies sufficientarianism (Bossert, Cato, and Kamaga, 2022, 2023) to each state. A social ordering R is *ex-post sufficientarian* if and only if there exists a continuous and increasing function $g\colon \mathbb{R} \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}$, uRv if and only if

$$\sum_{s \in S} \pi^s \sum_{i \in L(u^s)} (g(u_i^s) - g(\theta)) > \sum_{s \in S} \pi^s \sum_{i \in L(v^s)} (g(v_i^s) - g(\theta))$$

or

$$\sum_{s \in S} \pi^s \sum_{i \in L(u^s)} (g(u_i^s) - g(\theta)) = \sum_{s \in S} \pi^s \sum_{i \in L(v^s)} (g(v_i^s) - g(\theta)) \text{ and}$$
$$\sum_{s \in S} \pi^s \sum_{i \in H(u^s)} (g(u_i^s) - g(\theta)) \geq \sum_{s \in S} \pi^s \sum_{i \in H(v^s)} (g(v_i^s) - g(\theta)).$$

We note that if there is no risk (that is, if $u, v \in \mathcal{D}^c$), any ex-post sufficientarian ordering coincides with the corresponding sufficientarian ordering proposed by Bossert, Cato, and Kamaga (2022).

3 Basic axioms

First, we introduce the strong Pareto principle defined for the evaluation of riskless prospects.

Strong Pareto for no risk: For all $u, v \in \mathcal{D}^c$, if $u_i^s \geq v_i^s$ for all $i \in N$ and $u_i^s > v_i^s$ for some $i \in N$, then uPv.

The continuity axiom is a robustness condition. It requires that small changes in a prospect do not lead to large changes in the social ordering.

Continuity: For all $u \in \mathcal{D}$, the sets $\{v \in \mathcal{D} \mid vRu\}$ and $\{v \in \mathcal{D} \mid uRv\}$ are closed in $\mathcal{D}$.

Anonymity is an uncontroversial and fundamental impartiality property. It requires that all individuals' ex-post utilities be treated equally.

Anonymity: For all $u, v \in \mathcal{D}$ and for all bijections $\rho\colon N \to N$, if $v_i = u_{\rho(i)}$ for all $i \in N$, then uIv.

Our next axiom requires that a social ordering satisfy the expected-utility hypothesis. More precisely, we assume that there exists a social von Neumann-Morgenstern function W such that prospects are ranked by the comparison of the expected values of ex-post social welfare.

Social expected-utility hypothesis: There exists a function $W\colon \mathbb{R}^n \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}$,

$$uRv \;\Leftrightarrow\; \sum_{s \in S} \pi^s W(u^s) \geq \sum_{s \in S} \pi^s W(v^s).$$

The social expected-utility hypothesis implies statewise dominance, a property that is familiar from the literature on decision theory.

Statewise dominance: For all $u, v \in \mathcal{D}$, if $[u^s]P[v^s]$ for all $s \in S$, then uPv.

Define, for all non-empty strict subsets M of N and for all $u \in \mathcal{D}$, $u_M = (u_i)_{i \in M}$ and $u_{N \setminus M} = (u_i)_{i \in N \setminus M}$. Using this notation, we now introduce an independence axiom that has considerable intuitive appeal. The condition requires that a social ordering be independent of the ex-post utilities of those who are unconcerned in every state.

Prospect independence of the unconcerned: For all $u, u', v, v' \in \mathcal{D}$ and for all $\emptyset \neq M \subsetneq N$,

$$(u_M, v_{N \setminus M}) R (u'_M, v_{N \setminus M}) \;\Leftrightarrow\; (u_M, v'_{N \setminus M}) R (u'_M, v'_{N \setminus M}).$$

Independence properties of this nature are ubiquitous not only in the literature on social evaluation but, more generally, in numerous approaches in economics and political philosophy. The underlying intuition is very transparent and allows for a powerful defense of the requirement. In the statement of the axiom, those in $N \setminus M$ are unconcerned—the choice of prospects to be compared does not affect their ex-post utilities. It seems only natural that the resulting comparisons do not depend on these utility levels. That this separability property is highly plausible becomes apparent especially if a comprehensive notion of who is included in the overall population N is employed. It is usually assumed (at least implicitly) that a utility distribution (or, in our case, a prospect) represents a full history, from the remote past to the distant future, of the lifetime well-being of those who ever live. This includes individuals whose lives are long over, such as Cleopatra or Aristotle—and, more importantly, less prominent persons about whose lives very little (if anything) is known. If a comparison of two prospects were to depend on the ex-post utilities of the long dead, serious difficulties could not but emerge immediately: in the absence of knowledge regarding the utilities in question, the ex-post utilities of those who are affected may be affected dramatically. Another example appears in Blackorby, Bossert, and Donaldson (2005, pp. 132–133) as part of their defense of the use of independence axioms in the context of population ethics. Suppose that, in the not-too-distant future, a group of individuals departs Earth on a spaceship to settle, after several generations, on a planet in a distant star system. Those who leave lose all contact with those who remain on Earth, and the two groups will never hear from each other again. Properties such as prospect independence of the unconcerned ensure that decisions taken by the colonists do not depend on those who remain—a conclusion that has strong intuitive appeal. Because we work within a fixed overall population in this paper, prospect independence of the unconcerned is the only primary separability condition considered here. There are several versions of separability in our fixed-population setting because of risks, but this version is considered to be plausible for examining ex-post welfare criteria; Adler (2022, pp. 66–75) introduces prospect independence of the unconcerned under the name of "policy separability" and offers its normative defense in detail. In a variable-population setting, additional versions that are just as plausible can be considered; see, again, Blackorby, Bossert, and Donaldson (2005, Chapter 5) for a detailed discussion.

We note that Fleurbaey and Zuber (2013) use a similar separability property which they label independence of the utilities of the sure. This property restricts $v_{N\setminus M}$ and $v'_{N\setminus M}$ to those whose utility levels are constant across states. Thus, their condition (which is stated formally in the Appendix) states that a social ranking is independent of the ex-post utilities of those who are unconcerned in every state and bear no risk. This property is obviously weaker than our condition. Notably, independence of the utilities of the sure is not enough to establish our main characterization results; see the Appendix for counterexamples.

Finally, we present a version of the Pigou-Dalton transfer principle (Pigou, 1912; Dalton, 1920), which formalizes an equity consideration. The variant that we employ merely requires that a progressive transfer is desirable for prospects with no risk.

Pigou-Dalton transfer principle for no risk: For all $u, v \in \mathcal{D}^c$, if there exist $i, j \in N$ and $\delta \in \mathbb{R}_{++}$ such that $v_i^s = u_i^s - \delta \geq u_j^s + \delta = v_j^s$ and $u_k^s = v_k^s$ for all $k \in N \setminus \{i, j\}$, then vPu.

4 Ex-post prioritarianism

This section characterizes a class of ex-post prioritarian orderings. We begin with the following lemma, which is restricted to prospects with no risk.

Lemma 1. If a social ordering R satisfies strong Pareto for no risk, continuity, anonymity, and prospect independence of the unconcerned, then there exists a continuous and increasing function $g\colon \mathbb{R} \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}^c$,

$$uRv \;\Leftrightarrow\; \sum_{i\in N} g(u_i^s) \geq \sum_{i\in N} g(v_i^s).$$

Proof. Let $u, v \in \mathcal{D}^c$. Since u and v are riskless, letting $s \in S$, we can define the ordering R^s on $\mathbb{R}^n$ such that, for all $u, v \in \mathcal{D}^c$,

$$u^s R^s v^s \;\Leftrightarrow\; uRv.$$

Note that strong Pareto for no risk, anonymity, continuity, and prospect independence of the unconcerned imply that R^s satisfies the corresponding properties. Since $n \geq 3$, there exists a continuous and increasing function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$u^s R^s v^s \;\Leftrightarrow\; \sum_{i\in N} g(u_i^s) \geq \sum_{i\in N} g(v_i^s);$$

see Debreu (1959, pp. 56–59) and Blackorby, Bossert, and Donaldson (2005, Theorem 4.7). Combining these equivalences, the lemma is proved. ■

To present the next lemma, we need some additional notation and definitions. Given a continuous and increasing function $g\colon \mathbb{R} \to \mathbb{R}$, let Y denote the set of attainable values of the sum of transformed utilities $\sum_{i\in N} g(u_i^s)$ in state $s \in S$. The set Y is a non-degenerate open interval because g is continuous and increasing and $\mathbb{R}$ is connected.

We now show that a generalized class of ex-post criteria is obtained if the social expected-utility hypothesis is added to the axioms that appear in Lemma 1; see Theorem 3 of Blackorby, Bossert, and Donaldson (1998) for a related result that is established for a social ordering defined on lotteries.

Lemma 2. If a social ordering R satisfies strong Pareto for no risk, continuity, anonymity, the social expected-utility hypothesis, and prospect independence of the unconcerned, then there exist continuous and increasing functions $g\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon Y \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}$,

$$uRv \;\Leftrightarrow\; \sum_{s\in S} \pi^s \psi\left(\sum_{i\in N} g(u_i^s)\right) \geq \sum_{s\in S} \pi^s \psi\left(\sum_{i\in N} g(v_i^s)\right).$$

Proof. By the social expected-utility hypothesis, there exists a function $W\colon \mathbb{R}^n \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}$,

$$uRv \;\Leftrightarrow\; \sum_{s\in S} \pi^s W(u^s) \geq \sum_{s\in S} \pi^s W(v^s). \tag{1}$$

Since $\mathcal{D}^c \subset \mathcal{D}$, it follows that, for all $u, v \in \mathcal{D}^c$,

$$uRv \;\Leftrightarrow\; W(u^s) \geq W(v^s). \tag{2}$$

Since R satisfies continuity, W can be chosen to be continuous. Lemma 1 implies that there exists a continuous and increasing function $g\colon \mathbb{R} \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}^c$,

$$uRv \;\Leftrightarrow\; \sum_{i\in N} g(u_i^s) \geq \sum_{i\in N} g(v_i^s). \tag{3}$$

From (2) and (3), we obtain that, for all $u^s = (u_1^s, \ldots, u_n^s), v^s = (v_1^s, \ldots, v_n^s) \in \mathbb{R}^n$,

$$W(u^s) \geq W(v^s) \;\Leftrightarrow\; \sum_{i\in N} g(u_i^s) \geq \sum_{i\in N} g(v_i^s).$$

Therefore, there exists an increasing function $\psi\colon Y \to \mathbb{R}$ such that, for all $u^s = (u_1^s, \ldots, u_n^s) \in \mathbb{R}^n$,

$$W(u^s) = \psi\left(\sum_{i\in N} g(u_i^s)\right).$$

Since W is continuous, ψ can be chosen to be continuous. By (1), the lemma is proved. ■

In the following theorem, we provide a characterization of ex-post generalized utilitarianism using the axioms of Lemma 2. As its proof shows, the function ψ that appears in the statement of Lemma 2 must be affine in the presence of prospect independence of the unconcerned.

Theorem 1. A social ordering R satisfies strong Pareto for no risk, continuity, anonymity, the social expected-utility hypothesis, and prospect independence of the unconcerned if and only if R is an ex-post generalized utilitarian ordering.

Proof. It is straightforward to prove the 'if' part of the theorem statement. To prove the 'only if' part, observe first that Lemma 2 implies the existence of continuous and increasing functions $g\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon Y \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}$,

$$uRv \;\Leftrightarrow\; \sum_{s\in S} \pi^s \psi\left(\sum_{i\in N} g(u_i^s)\right) \geq \sum_{s\in S} \pi^s \psi\left(\sum_{i\in N} g(v_i^s)\right). \tag{4}$$

To show that ψ is affine, let $(\gamma^1, \gamma^2) \in Y^2$. Since ψ is continuous and increasing on Y, there exists $(\tilde{\gamma}^1, \tilde{\gamma}^2) \in Y^2$ with $\gamma^1 > \tilde{\gamma}^1$ and $\gamma^2 < \tilde{\gamma}^2$ such that

$$\sum_{s=1}^{2} \pi^s \psi(\gamma^s) = \sum_{s=1}^{2} \pi^s \psi(\tilde{\gamma}^s). \tag{5}$$

Step 1. Assume first that n is even. We show that, for any $a \in (0, 1)$,

$$\sum_{s=1}^{2} \pi^s \psi(\gamma^s) = \sum_{s=1}^{2} \pi^s \psi(a\gamma^s + (1-a)\tilde{\gamma}^s). \tag{6}$$

Let $(\bar{\gamma}^1, \bar{\gamma}^2)$ denote the midpoint of (γ^1, γ^2) and $(\tilde{\gamma}^1, \tilde{\gamma}^2)$ in Y^2. Formally, for each $s = 1, 2$,

$$\bar{\gamma}^s = \frac{\gamma^s + \tilde{\gamma}^s}{2}.$$

We begin by showing that

$$\sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s) = \sum_{s=1}^{2} \pi^s \psi(\gamma^s). \tag{7}$$

Since $(\bar{\gamma}^1, \bar{\gamma}^2) \in Y^2$, there exist $u, v \in \mathcal{D}$ such that, for each $s = 1, 2$,

$$\begin{aligned}
&u_i^s = g^{-1}(\gamma^s/n) \;\text{ and }\; v_i^s = g^{-1}(\tilde{\gamma}^s/n) \text{ for all } i \in \{1, \ldots, n/2\},\\
&u_j^s = v_j^s = g^{-1}(\gamma^s/n) \text{ for all } j \in \{n/2+1, \ldots, n\},
\end{aligned}$$

and $u_i^s = v_i^s$ for all $s \in S \setminus \{1, 2\}$ and for all $i \in N$. Furthermore, there exist $\hat{u}, \hat{v} \in \mathcal{D}$ such that, for each $s = 1, 2$,

$$\begin{aligned}
&u_i^s = \hat{u}_i^s \text{ and } v_i^s = \hat{v}_i^s \text{ for all } i \in \{1, \ldots, n/2\},\\
&\hat{u}_j^s = \hat{v}_j^s = g^{-1}(\tilde{\gamma}^s/n) \text{ for all } j \in \{n/2+1, \ldots, n\},
\end{aligned}$$

and $\hat{u}_i^s = \hat{v}_i^s = u_i^s$ for all $s \in S \setminus \{1,2\}$ and for all $i \in N$. Note that, for each $s = 1, 2$,

$$\sum_{i \in N} g(u_i^s) = n \cdot \frac{\gamma^s}{n} = \gamma^s, \;\; \sum_{i \in N} g(v_i^s) = \sum_{i \in N} g(\hat{u}_i^s) = \bar{\gamma}^s \;\text{ and }\; \sum_{i \in N} g(\hat{v}_i^s) = n \cdot \frac{\tilde{\gamma}^s}{n} = \tilde{\gamma}^s.$$

Since R satisfies prospect independence of the unconcerned, we obtain

$$uRv \;\Leftrightarrow\; \hat{u}R\hat{v} \;\text{ and }\; vRu \;\Leftrightarrow\; \hat{v}R\hat{u}.$$

Thus, if uPv holds, then $\hat{u}P\hat{v}$ follows and we obtain by Lemma 2 that

$$\sum_{s=1}^{2} \pi^s \psi(\gamma^s) > \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s) \;\text{ and }\; \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s) > \sum_{s=1}^{2} \pi^s \psi(\tilde{\gamma}^s),$$

and we obtain a contradiction to (5). Similarly, if vPu holds, it follows that

$$\sum_{s=1}^{2} \pi^s \psi(\gamma^s) < \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s) < \sum_{s=1}^{2} \pi^s \psi(\tilde{\gamma}^s),$$

a contradiction. Hence, uIv must hold, and $\hat{u}I\hat{v}$ follows as well. Thus, by (4), we obtain (7). Since ψ is continuous, applying the above argument repeatedly, we obtain that (6) holds for any $a \in (0,1)$.

Step 2. Now suppose that n is odd. We show that (6) holds for any $a \in (0,1)$. For all $t \in \mathbb{N} \setminus \{1\}$ and for all $\ell \in \{1, \ldots, t-1\}$, define $\bar{\gamma}^s(\ell, t) \in \mathbb{R}$ by, for each $s = 1, 2$,

$$\bar{\gamma}^s(\ell, t) = \frac{\ell}{t}\tilde{\gamma}^s + \frac{t-\ell}{t}\gamma^s.$$

Note that, given $t \in \mathbb{N}$, for all $\ell \in \{2, \ldots, t-1\}$,

$$\lim_{a \to \infty} n \cdot g(a) > \gamma^1 > \bar{\gamma}^1(\ell-1, t) > \bar{\gamma}^1(\ell, t) \;\text{ and }\; \lim_{a \to -\infty} n \cdot g(a) < \gamma^2 < \bar{\gamma}^2(\ell-1, t) < \bar{\gamma}^2(\ell, t).$$

Thus, there exists $t^1 \in \mathbb{N} \setminus \{1\}$ such that, for all $t \geq t^1$, there exists $(u_1^1, \ldots, u_n^1) \in \mathbb{R}^n$ such that

$$\begin{aligned} &g(u_i^1) = g(u_j^1) > \frac{\gamma^1}{n} \text{ for all } i, j \in \{1, \ldots, n-1\}, \\ &g(u_n^1) = \frac{\bar{\gamma}^1(1,t)}{n} < \frac{\gamma^1}{n} \;\text{ and }\; \sum_{i \in N} g(u_i^1) = \gamma^1. \end{aligned}$$

Moreover, there exists $t^2 \in \mathbb{N} \setminus \{1\}$ such that, for all $t \geq t^2$, there exists $(u_1^2, \ldots, u_n^2) \in \mathbb{R}^n$ such that

$$\begin{aligned} &g(u_i^2) = g(u_j^2) < \frac{\gamma^2}{n} \text{ for all } i \in \{1, \ldots, n-1\}, \\ &g(u_n^2) = \frac{\bar{\gamma}^2(1,t)}{n} > \frac{\gamma^2}{n} \;\text{ and }\; \sum_{i \in N} g(u_i^2) = \gamma^2. \end{aligned}$$

We now define $t^* = 2 \cdot \max\{t^1, t^2\}$. Then, there exist $u, v \in \mathcal{D}$ such that, for each $s = 1, 2$,

$$\begin{aligned}
&u_i^s = u_j^s \text{ for all } i, j \in \{1, \ldots, n-1\}, \\
&g(u_n^s) = \frac{\bar{\gamma}^s(2, t^*)}{n} \text{ and } \sum_{i \in N} g(u_i^s) = \gamma^s, \\
&g(v_i^s) = \frac{\bar{\gamma}^s(2, t^*)}{n} \text{ for all } i \in \{1, \ldots, (n-1)/2\} \cup \{n\}, \\
&v_j^s = u_j^s \text{ for all } j \in \{(n-1)/2 + 1, \ldots, n-1\},
\end{aligned}$$

and $u_i^s = v_i^s$ for all $s \in S \setminus \{1, 2\}$ and for all $i \in N$. Furthermore, there exist $\hat{u}, \hat{v} \in \mathcal{D}$ such that, for each $s = 1, 2$,

$$\begin{aligned}
&\hat{u}_i^s = u_i^s \text{ and } \hat{v}_i^s = v_i^s \text{ for all } i \in \{1, \ldots, (n-1)/2\} \cup \{n\}, \\
&g(\hat{u}_j^s) = g(\hat{v}_j^s) = \frac{\bar{\gamma}^s(2, t^*)}{n} \text{ for all } j \in \{(n-1)/2 + 1, \ldots, n-1\},
\end{aligned}$$

and $\hat{u}_i^s = \hat{v}_i^s = u_i^s$ for all $s \in S \setminus \{1, 2\}$ and for all $i \in N$. Note that, for each $s = 1, 2$,

$$\begin{aligned}
\sum_{i \in N} g(v_i^s) &= \frac{n+1}{2} \cdot \frac{\bar{\gamma}^s(2, t^*)}{n} + \frac{n-1}{2} g(u_1^s) \\
&= \frac{n+1}{2} \cdot \frac{\bar{\gamma}^s(2, t^*)}{n} + \frac{n-1}{2} \cdot \frac{1}{n-1} \cdot \left(\gamma^s - \frac{\bar{\gamma}^s(2, t^*)}{n}\right) \\
&= \frac{\bar{\gamma}^s(2, t^*) + \gamma^s}{2} \\
&= \bar{\gamma}^s(1, t^*), \\
\sum_{i \in N} g(\hat{u}_i^s) &= \frac{n-1}{2} \cdot \frac{1}{n-1} \cdot \left(\gamma^s - \frac{\bar{\gamma}^s(2, t^*)}{n}\right) + \frac{n+1}{2} \cdot \frac{\bar{\gamma}^s(2, t^*)}{n} \\
&= \bar{\gamma}^s(1, t^*),
\end{aligned}$$

and

$$\sum_{i \in N} g(\hat{v}_i^s) = \bar{\gamma}^s(2, t^*).$$

Since R satisfies prospect independence of the unconcerned, applying the argument employed in Step 1, we obtain the following three cases.

$$\begin{aligned}
&\text{(a)} \sum_{s=1}^{2} \pi^s \psi(\gamma^s) > \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(1, t^*)) > \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(2, t^*)), \\
&\text{(b)} \sum_{s=1}^{2} \pi^s \psi(\gamma^s) < \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(1, t^*)) < \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(2, t^*)), \\
&\text{(c)} \sum_{s=1}^{2} \pi^s \psi(\gamma^s) = \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(1, t^*)) = \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(2, t^*)).
\end{aligned}$$

We show by contradiction that case (c) holds. First, suppose that case (a) holds. Then, we can find $u, v \in \mathcal{D}$ such that, for each $s = 1, 2$,

$$\begin{aligned}
&u_i^s = u_j^s \text{ for all } i, j \in \{1, \dots, n-1\}, \\
&g(u_n^s) = \frac{\bar{\gamma}^s(3, t^*)}{n} \text{ and } \sum_{i \in N} g(u_i^s) = \bar{\gamma}^s(1, t^*), \\
&g(v_i^s) = \frac{\bar{\gamma}^s(3, t^*)}{n} \text{ for all } i \in \{1, \dots, (n-1)/2\} \cup \{n\}, \\
&v_j^s = u_j^s \text{ for all } j \in \{(n-1)/2 + 1, \dots, n-1\},
\end{aligned}$$

and $u_i^s = v_i^s$ for all $s \in S \setminus \{1, 2\}$ and for all $i \in N$. Furthermore, there exist $\hat{u}, \hat{v} \in \mathcal{D}$ such that, for each $s = 1, 2$,

$$\begin{aligned}
&\hat{u}_i^s = u_i^s \text{ and } \hat{v}_i^s = v_i^s \text{ for all } i \in \{1, \dots, (n-1)/2\} \cup \{n\}, \\
&g(\hat{u}_j^s) = g(\hat{v}_j^s) = \frac{\bar{\gamma}^s(3, t^*)}{n} \text{ for all } j \in \{(n-1)/2 + 1, \dots, n-1\},
\end{aligned}$$

and $\hat{u}_i^s = \hat{v}_i^s = u_i^s$ for all $s \in S \setminus \{1, 2\}$ and for all $i \in N$. Note that, for each $s = 1, 2$,

$$\sum_{i \in N} g(v_i^s) = \bar{\gamma}^s(2, t^*) = \sum_{i \in N} g(\hat{u}_i^s) \text{ and } \sum_{i \in N} g(\hat{v}_i^s) = \bar{\gamma}^s(3, t^*).$$

Thus, it follows from (4) and prospect independence of the unconcerned that

$$\sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(1, t^*)) > \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(2, t^*)) \Rightarrow \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(2, t^*)) > \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(3, t^*)).$$

Applying this argument repeatedly, we obtain that

$$\sum_{s=1}^{2} \pi^s \psi(\gamma^s) > \sum_{s=1}^{2} \pi^s \psi(\tilde{\gamma}^s).$$

However, this is a contradiction to (5). Similarly, if case (b) holds, we obtain a contradiction. Therefore, case (c) must hold.

Applying the argument that we used to show a contradiction in case (a), we obtain that, for each $\ell \in \{2, \dots, t^* - 1\}$,

$$\sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(\ell - 1, t^*)) = \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s(\ell, t^*))$$

Since ψ is continuous, it follows from the same argument as in Step 1 that (6) holds for any $a \in (0, 1)$.

Step 3. Applying the argument used to derive the implication of case (c) in Step 2, we can extend the result that (6) holds for any $a \in (0, 1)$ to any parameter $a \in (-\infty, 0) \cup (1, \infty)$. Therefore, we can conclude that, for any $(\bar{\gamma}^1, \bar{\gamma}^2) \in Y^2$ that lies on the straight line passing through (γ^1, γ^2) and $(\tilde{\gamma}^1, \tilde{\gamma}^2)$,

$$\sum_{s=1}^{2} \pi^s \psi(\gamma^s) = \sum_{s=1}^{2} \pi^s \psi(\bar{\gamma}^s).$$

This implies that there exists $(\alpha^1, \alpha^2) \in \mathbb{R}^2_{++}$ such that, for all $(\gamma^1, \gamma^2) \in Y^2$,

$$\sum_{s=1}^{2} \pi^s \psi(\gamma^s) = \sum_{s=1}^{2} \alpha^s \gamma^s.$$

Thus, given a fixed $\gamma^2 = \bar{\gamma}^2$, it follows that, for all $\gamma^1 \in Y$,

$$\psi(\gamma^1) = \frac{\alpha^1}{\pi^1}\gamma^1 + \frac{\alpha^2\bar{\gamma}^2 - \pi^2\psi(\bar{\gamma}^2)}{\pi^1}.$$

Therefore, ψ is affine. Consequently, we can assume that ψ in (4) is given by $\psi(a) = a$ for all $a \in Y$. ■

If we require the Pigou-Dalton transfer principle for no risk in addition to the axioms of Theorem 1, the utility transformation g that ex-post generalized utilitarianism employs must be strictly midpoint-concave (that is, $g((x+y)/2) > [g(x) + g(y)]/2$ for all $x, y \in \mathbb{R}$ with $x \neq y$). Since any continuous, strictly midpoint-concave function is strictly concave, only the class of ex-post prioritarian orderings is permissible. We first state a variant of Lemma 1 of Fleurbaey and Zuber (2013), which shows that the conjunction of the axioms used in our characterization of ex-post prioritarianism implies anonymity. The axioms of Fleurbaey and Zuber's (2013) lemma are slightly different from ours; to be precise, the social rationality and independence axioms that they employ are weaker than ours but their transfer axiom is stronger than ours. However, as they state in their discussion (Fleurbaey and Zuber, 2013, p. 685), the Pigou–Dalton transfer principle for no risk suffices to prove their lemma. Thus, we state the following lemma without a proof.

Lemma 3. If a social ordering R satisfies strong Pareto for no risk, continuity, the social expected-utility hypothesis, prospect independence of the unconcerned, and the Pigou-Dalton transfer principle for no risk, then R satisfies anonymity.

The following theorem axiomatizes ex-post prioritarianism.

Theorem 2. A social ordering R satisfies strong Pareto for no risk, continuity, the social expected-utility hypothesis, prospect independence of the unconcerned, and the Pigou-Dalton transfer principle for no risk if and only if R is an ex-post prioritarian ordering.

Proof. 'If.' Suppose that R is an ex-post prioritarian ordering. This implies that R is ex-post generalized utilitarian and, by Theorem 1, R satisfies all axioms other than the Pigou-Dalton transfer principle for no risk. It is easy to show that the Pigou-Dalton transfer principle for no risk is also satisfied; see, for example, Table 4.2 of Blackorby, Bossert, and Donaldson (2005, p. 82).

'Only if.' Assume that a social ordering R satisfies strong Pareto for no risk, continuity, the social expected-utility hypothesis, prospect independence of the unconcerned, and the Pigou-Dalton transfer principle for no risk. By Lemma 3, R satisfies anonymity. Theorem 1 implies that R is ex-post generalized utilitarian. As mentioned above, g must be strictly concave because of the Pigou-Dalton transfer principle for no risk. Thus, R must be ex-post prioritarian. ■

In the above theorem, the social expected-utility hypothesis cannot be weakened to the following alternative social rationality axiom.

Sensitivity to risk: For all $u, v \in \mathcal{D}$, if there exist $s, s' \in S$ with $\pi_s > \pi_{s'}$ such that

$$[u^s]P[v^s],\ u^{s'} = v^s,\ v^{s'} = u^s,\ \text{and } u^t = v^t \text{ for all } t \in S \setminus \{s, s'\},$$

then uPv.

A social ordering R is *ex-post prioritarian with probability weighing* if and only if there exist an increasing and strictly concave function $g\colon \mathbb{R} \to \mathbb{R}$ and an increasing and continuous function $\phi\colon (0,1) \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}$,

$$uRv \;\Leftrightarrow\; \sum_{s \in S} \phi(\pi^s) \sum_{i \in N} g(u_i^s) \geq \sum_{s \in S} \phi(\pi^s) \sum_{i \in N} g(v_i^s).$$

This ordering satisfies strong Pareto for no risk, continuity, anonymity, statewise dominance, prospect independence of the unconcerned, the Pigou-Dalton transfer principle for no risk, and sensitivity to risk. Unless the function ϕ is a homogeneous linear function, these orderings do not satisfy the social expected-utility hypothesis.

5 Ex-post sufficientarianism

According to sufficientarianism, absolute priority is assigned to utility levels below the sufficiency threshold θ. This suggests that sufficientarian theories primarily care about changes below the threshold. Thus, as an auxiliary step, it is helpful to introduce censored profiles at the level of the threshold θ. For each $u \in \mathcal{D}$, let

$$u_L = (\min\{u_i^s, \theta\})_{i \in N, s \in S},$$

and

$$u_H = (\max\{u_i^s, \theta\})_{i \in N, s \in S}.$$

Typically, when considering two prospects u and v, sufficientarian orderings first compare u_L and v_L. If required, a comparison between u_H and v_H is employed as a tie-breaking criterion. The idea that absolute priority should be given to those below the threshold constitutes the core of sufficientarianism; see, for example, Crisp (2003), Brown (2005), and Casal (2007). The following axiom is a natural extension of the fundamental property of sufficientarianism to the evaluation of prospects.

Ex-post absolute priority: For all $u, v \in \mathcal{D}$,

$$u_L P v_L \;\Rightarrow\; uPv$$

and

$$u_L I v_L \;\Rightarrow\; [uRv \Leftrightarrow u_H R v_H]\,.$$

Ex-post absolute priority and the social expected-utility hypothesis together are incompatible with strong Pareto for no risk. This impossibility, stated in the following theorem, arises as a consequence of the lexical treatment embodied by ex-post absolute priority.

Theorem 3. There exists no social ordering R that satisfies strong Pareto for no risk, the social expected utility hypothesis, and ex-post absolute priority.

Proof. Suppose that $n = 3$ and that the social ordering R satisfies strong Pareto for no risk, the social expected utility hypothesis, and ex-post absolute priority. For all $a \in [\theta - 1, \theta]$, let $u(a)$ and $v(a)$ be the riskless prospects in $\mathcal{D}^c$ defined by letting, for all $s \in S$,

$$u(a)_1^s = a, \; u(a)_2^s = u(a)_3^s = \theta + 1$$

and

$$v(a)_1^s = a, \; v(a)_2^s = v(a)_3^s = \theta + 2.$$

Then, for all $a, b \in [\theta - 1, \theta]$ with $a > b$, the three axioms together imply that $W(u(a)^s) > W(v(b)^s) > W(u(b)^s)$. Therefore, the non-degenerate intervals

$$I(a) = [W(u(a)^s), W(v(a)^s)] \;\text{ and }\; I(b) = [W(u(b)^s), W(v(b)^s)]$$

are mutually disjoint. Since the interval $[\theta - 1, \theta]$ is uncountable, each of uncountably many intervals $I(a)$ contains a rational number. This is a contradiction because rational numbers are countable. ■

In view of Theorem 3, we need to weaken the social expected-utility hypothesis if we are to respect the fundamental axiom of sufficientarianism. To do so, we define two subdomains of $\mathcal{D}$. Let

$$\mathcal{D}_L = \{u \in \mathcal{D} \mid u_i^s \leq \theta \text{ for all } i \in N \text{ and for all } s \in S\}$$

and

$$\mathcal{D}_H = \{u \in \mathcal{D} \mid u_i^s \geq \theta \text{ for all } i \in N \text{ and for all } s \in S\}.$$

Note that, for any $u \in \mathcal{D}$, $u_L \in \mathcal{D}_L$ and $u_H \in \mathcal{D}_H$.

The following axiom restricts the social expected-utility hypothesis to censored distributions.

Restricted social expected-utility hypothesis: There exists a function $W: \mathbb{R}^n \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}_L$,

$$uRv \;\Leftrightarrow\; \sum_{s \in S} \pi^s W(u^s) \geq \sum_{s \in S} \pi^s W(v^s)$$

and, for all $u, v \in \mathcal{D}_H$,

$$uRv \;\Leftrightarrow\; \sum_{s \in S} \pi^s W(u^s) \geq \sum_{s \in S} \pi^s W(v^s).$$

The conjunction of ex-post absolute priority and the restricted social expected-utility hypothesis can equivalently be represented by a single concise requirement stated as the following axiom.

Expected sufficientarian hypothesis: There exists a function $W: \mathbb{R}^n \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}$,

$$\sum_{s \in S} \pi^s W(u_L^s) > \sum_{s \in S} \pi^s W(v_L^s) \;\Rightarrow\; uPv$$

and

$$\sum_{s \in S} \pi^s W(u_L^s) = \sum_{s \in S} \pi^s W(v_L^s) \;\Rightarrow\; \left[\sum_{s \in S} \pi^s W(u_H^s) \geq \sum_{s \in S} \pi^s W(v_H^s) \;\Leftrightarrow\; uRv\right].$$

Lemma 4. A social ordering R satisfies the expected sufficientarian hypothesis if and only if R satisfies ex-post absolute priority and the restricted social expected-utility hypothesis.

Proof. 'Only if.' Assume that R satisfies the expected sufficientarian hypothesis. First, we show that the restricted social expected-utility hypothesis is satisfied. Let W be a function that satisfies the requisite property stated in the expected sufficientarian hypothesis. Let $u, v \in \mathcal{D}_L$. Note that $u = u_L$ and $v = v_L$. Thus, the expected sufficientarian hypothesis implies that

$$\sum_{s \in S} \pi^s W(u^s) > \sum_{s \in S} \pi^s W(v^s) \;\Rightarrow\; uPv.$$

Because u_H and v_H are empty in this case, the equality $\sum_{s \in S} \pi^s W(u^s) = \sum_{s \in S} \pi^s W(v^s)$ implies uIv. Therefore,

$$\sum_{s \in S} \pi^s W(u^s) \geq \sum_{s \in S} \pi^s W(v^s) \;\Leftrightarrow\; uRv.$$

The proof of the second part of the restricted social expected-utility hypothesis is analogous.

Next, we show that ex-post absolute priority is satisfied. Let $u, v \in \mathcal{D}$. As shown above, the expected sufficientarian hypothesis implies the restricted social expected-utility hypothesis. Thus, if $u_L P v_L$, then

$$\sum_{s \in S} \pi^s W(u_L^s) > \sum_{s \in S} \pi^s W(v_L^s),$$

since $u_L, v_L \in \mathcal{D}_L$. Now the expected sufficientarian hypothesis implies that uPv. Similarly, if $u_L I v_L$, it follows that

$$\sum_{s \in S} \pi^s W(u_L^s) = \sum_{s \in S} \pi^s W(v_L^s).$$

The expected sufficientarian hypothesis implies

$$\sum_{s \in S} \pi^s W(u_H^s) \geq \sum_{s \in S} \pi^s W(v_H^s) \;\Leftrightarrow\; uRv.$$

By the restricted social expected-utility hypothesis,

$$u_H R v_H \;\Leftrightarrow\; \sum_{s \in S} \pi^s W(u_H^s) \geq \sum_{s \in S} \pi^s W(v_H^s).$$

Thus, combining these equivalences, we obtain

$$u_H R v_H \;\Leftrightarrow\; uRv$$

so that R satisfies ex-post absolute priority.

'If.' Suppose that R satisfies ex-post absolute priority and the restricted social expected-utility hypothesis. Let W be a function that satisfies the requisite property stated in the restricted social expected-utility hypothesis. Let $u, v \in \mathcal{D}$. First, assume that

$$\sum_{s \in S} \pi^s W(u_L^s) > \sum_{s \in S} \pi^s W(v_L^s).$$

Note that $u_L, v_L \in \mathcal{D}_L$. Thus, it follows from the restricted social expected-utility hypothesis that

$$u_L P v_L.$$

From ex-post absolute priority, uPv follows. Next, we assume that

$$\sum_{s \in S} \pi^s W(u_L^s) = \sum_{s \in S} \pi^s W(v_L^s).$$

The restricted social expected-utility hypothesis implies that $u_L I v_L$. By ex-post absolute priority, we obtain

$$u_H R v_H \;\Leftrightarrow\; uRv.$$

From the restricted social expected-utility hypothesis, it follows that

$$\sum_{s \in S} \pi^s W(u_H^s) \geq \sum_{s \in S} \pi^s W(v_H^s) \;\Leftrightarrow\; u_H R v_H.$$

Thus, combining these equivalences, we obtain

$$\sum_{s \in S} \pi^s W(u_H^s) \geq \sum_{s \in S} \pi^s W(v_H^s) \;\Leftrightarrow\; uRv.$$

Therefore, R satisfies the expected sufficientarian hypothesis. ■

As pointed out by Roemer (2004), sufficientarianism cannot be entirely continuous, but it is conditionally continuous; see Bossert, Cato, and Kamaga (2022, 2023) for a discussion of this issue in a deterministic setting. The following conditional continuity axioms require that R be continuous on the subdomains $\mathcal{D}_L$ and $\mathcal{D}_H$, respectively.

Continuity below the threshold: For all $u \in \mathcal{D}_L$, the sets $\{v \in \mathcal{D}_L \mid vRu\}$ and $\{v \in \mathcal{D}_L \mid uRv\}$ are closed in $\mathcal{D}_L$.

Continuity above the threshold: For all $u \in \mathcal{D}_H$, the sets $\{v \in \mathcal{D}_H \mid vRu\}$ and $\{v \in \mathcal{D}_H \mid uRv\}$ are closed in $\mathcal{D}_H$.

Ex-post sufficientarianism is characterized by replacing the social expected-utility hypothesis and continuity in Theorem 1 with the expected sufficientarian hypothesis and the two conditional continuity axioms.

Theorem 4. A social ordering R satisfies strong Pareto for no risk, continuity above the threshold, continuity below the threshold, anonymity, the expected sufficientarian hypothesis, and prospect independence of the unconcerned if and only if R is an ex-post sufficientarian ordering.

We begin with two lemmas using Theorem 1. The first of these states that ex-post generalized utilitarianism must be applied to prospects below the threshold.

Lemma 5. If a social ordering R satisfies strong Pareto for no risk, continuity below the threshold, anonymity, the expected sufficientarian hypothesis, and prospect independence of the unconcerned, then there exists a continuous and increasing function $g_L \colon (-\infty, \theta] \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}_L$,

$$uRv \;\Leftrightarrow\; \sum_{s \in S} \pi^s \sum_{i \in N} g_L(u_i^s) \geq \sum_{s \in S} \pi^s \sum_{i \in N} g_L(v_i^s). \tag{8}$$

Proof. Suppose that R satisfies strong Pareto for no risk, continuity below the threshold, anonymity, the expected sufficientarian hypothesis, and prospect independence of the unconcerned. By Lemma 4, R satisfies the restricted social expected-utility hypothesis. Let R_L be the restriction of R on $\mathcal{D}_L$, that is, for all $u, v \in \mathcal{D}_L$,

$$uR_Lv \Leftrightarrow uRv.$$

Note that R_L satisfies strong Pareto for no risk, continuity, anonymity, prospect independence of the unconcerned, and the social expected-utility hypothesis on $\mathcal{D}_L$. Applying Theorem 1 to R_L, there exists a continuous and increasing function $g_L\colon (-\infty, \theta] \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}_L$,

$$uR_Lv \Leftrightarrow \sum_{s\in S} \pi^s \sum_{i\in N} g_L(u_i^s) \geq \sum_{s\in S} \pi^s \sum_{i\in N} g_L(v_i^s).$$

This establishes the statement of the lemma. ■

The next lemma states an analogous result for prospects above the threshold. Its proof is analogous to that of Lemma 5.

Lemma 6. If a social ordering R satisfies strong Pareto for no risk, continuity above the threshold, anonymity, the expected sufficientarian hypothesis, and prospect independence of the unconcerned, then there exists a continuous and increasing function $g_H\colon [\theta, \infty) \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}_H$,

$$uRv \Leftrightarrow \sum_{s\in S} \pi^s \sum_{i\in N} g_H(u_i^s) \geq \sum_{s\in S} \pi^s \sum_{i\in N} g_H(v_i^s). \tag{9}$$

Proof of Theorem 4. It is straightforward to verify that all axioms are satisfied by any ex-post sufficientarian ordering.

Conversely, suppose that R satisfies the axioms of the theorem statement. Lemma 4 implies that R satisfies ex-post absolute priority. From Lemmas 5 and 6, it follows that there exist continuous and increasing functions $g_L\colon (-\infty, \theta] \to \mathbb{R}$ and $g_H\colon [\theta, \infty) \to \mathbb{R}$ that satisfy (8) and (9), respectively. Define the function $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(a) = \begin{cases} g_L(a) - g_L(\theta) + g_H(\theta) & \text{if } a \in (-\infty, \theta) \\ g_H(a) & \text{if } a \in [\theta, +\infty). \end{cases}$$

This function is continuous and increasing on $\mathbb{R}$. We show that R is the ex-post sufficientarian ordering associated with g. Let $u, v \in \mathcal{D}$. We first assume that

$$\sum_{s\in S} \pi^s \sum_{i\in L(u^s)} (g(u_i^s) - g(\theta)) > \sum_{s\in S} \pi^s \sum_{i\in L(v^s)} (g(v_i^s) - g(\theta)).$$

Letting $u_L = w$ and $v_L = z$, this implies that

$$\sum_{s\in S} \pi^s \sum_{i\in N} g_L(w_i^s) > \sum_{s\in S} \pi^s \sum_{i\in N} g_L(z_i^s).$$

From Lemma 5, we obtain $u_L P v_L$. By ex-post absolute priority (which is implied by the expected sufficientarian hypothesis; see Lemma 4), uPv follows.

Next, we assume that

$$\sum_{s\in S}\pi^s \sum_{i\in L(u^s)} (g(u_i^s) - g(\theta)) = \sum_{s\in S}\pi^s \sum_{i\in L(v^s)} (g(v_i^s) - g(\theta)) \text{ and}$$
$$\sum_{s\in S}\pi^s \sum_{i\in H(u^s)} (g(u_i^s) - g(\theta)) \geq \sum_{s\in S}\pi^s \sum_{i\in H(v^s)} (g(v_i^s) - g(\theta)).$$

Applying an analogous argument to u_L, v_L, u_H, and v_H, it follows from Lemmas 5 and 6 that $u_L I v_L$ and $u_H R v_H$. By ex-post absolute priority, uRv follows. ■

From Lemma 4, we obtain the following corollary to Theorem 4.

Corollary 1. A social ordering R satisfies strong Pareto for no risk, continuity above the threshold, continuity below the threshold, anonymity, the restricted social expected-utility hypothesis, prospect independence of the unconcerned, and ex-post absolute priority if and only if R is ex-post sufficientarian.

The Pigou-Dalton transfer principle for no risk can be amended in the context of ex-post sufficientarian orderings. As is the case for the version used to characterize ex-post prioritarianism, it is sufficient to restrict attention to prospects with no risk. In analogy to the approach followed in Bossert, Cato, and Kamaga (2022, Section V), two versions of the Pigou-Dalton transfer principles for no risk can be employed—one that applies below the threshold, one that is defined above θ. If these two principles are added to the axioms of Theorem 4 (or of Corollary 1), the restrictions of the transformation g to utility values less than or equal to θ and to utility values greater than or equal to θ are strictly concave. Note, however, that this does not imply the strict concavity of g on its entire domain. See Bossert, Cato, and Kamaga (2022, Section V) for a detailed discussion.

6 Weak Pareto for equal risk

Our axiomatizations of ex-post prioritarianism and ex-post sufficientarianism employ prospect independence of the unconcerned. Although the members of these classes satisfy strong Pareto for no risk, none of them satisfy the requirement of weak Pareto for equal risk that Fleurbaey (2010) employs. To define this axiom, we consider another subdomain of $\mathcal{D}$. A prospect $u \in \mathcal{D}$ is *egalitarian* if $u_i = u_j$ for all $i, j \in N$. Let $\mathcal{D}^e$ be the set of egalitarian prospects. Note that, if $u \in \mathcal{D}^e$, then $E(u_i) = E(u_j)$ for all $i, j \in N$. Weak Pareto for equal risk postulates the weak Pareto principle for egalitarian prospects.

Weak Pareto for equal risk: For all $u, v \in \mathcal{D}^e$, if $E(u_i) > E(v_i)$ for all $i \in N$, then uPv.

Weak Pareto for equal risk by itself is compatible with prospect independence of the unconcerned. An example is the utilitarian ordering, which determines the social relation R by comparing the sums of the individuals' expected utilities; as is straightforward to verify, this ordering satisfies both prospect independence of the unconcerned and weak Pareto for equal risk. However, weak Pareto for equal risk cannot be satisfied by an ex-post prioritarian ordering. This impossibility result generalizes to the incompatibility between prospect independence of the unconcerned and weak Pareto for equal risk in the presence of continuity, anonymity, and the following variant of the Pigou-Dalton transfer principle.

Ex-post equalization principle: For all $u \in \mathcal{D}^e$ and for all $v \in \mathcal{D} \setminus \mathcal{D}^e$, if $\sum_{i \in N} u_i^s = \sum_{i \in N} v_i^s$ for all $s \in S$, then uPv.

As is straightforward to verify, the ex-post prioritarian orderings satisfy the ex-post equalization principle.

We obtain the following impossibility result.

Theorem 5. There exists no social ordering R that satisfies weak Pareto for equal risk, continuity, anonymity, prospect independence of the unconcerned, and the ex-post equalization principle.

Proof. For simplicity, we assume that there are two states and three individuals. Thus, we can use a 2×3 matrix to represent each prospect by writing it as

$$u = \begin{bmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \\ u_3^1 & u_3^2 \end{bmatrix}.$$

The first state occurs with probability p and the second with probability $1 - p$. By weak Pareto for equal risk and continuity, there exists a positive real number δ such that

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} I \begin{bmatrix} 0 & \delta \\ 0 & \delta \\ 0 & \delta \end{bmatrix}.$$

That is, $p = (1 - p)\delta$ holds. Define v as

$$v = \begin{bmatrix} 1 & 0 \\ 0 & \delta \\ 1 & 0 \end{bmatrix}.$$

By the ex-post equalization principle,

$$\begin{bmatrix} 2/3 & \delta/3 \\ 2/3 & \delta/3 \\ 2/3 & \delta/3 \end{bmatrix} P \begin{bmatrix} 1 & 0 \\ 0 & \delta \\ 1 & 0 \end{bmatrix}.$$

By continuity, there exists $\kappa > 0$ such that

$$\begin{bmatrix} 2/3 - \kappa & \delta/3 - \kappa \\ 2/3 - \kappa & \delta/3 - \kappa \\ 2/3 - \kappa & \delta/3 - \kappa \end{bmatrix} P \begin{bmatrix} 1 & 0 \\ 0 & \delta \\ 1 & 0 \end{bmatrix}.$$

By weak Pareto for equal risk,

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} P \begin{bmatrix} 2/3 - \kappa & \delta/3 - \kappa \\ 2/3 - \kappa & \delta/3 - \kappa \\ 2/3 - \kappa & \delta/3 - \kappa \end{bmatrix}.$$

By transitivity,

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} P \begin{bmatrix} 1 & 0 \\ 0 & \delta \\ 1 & 0 \end{bmatrix}.$$

Prospect independence of the unconcerned implies

$$\begin{bmatrix} 0 & \delta \\ 1 & 0 \\ 1 & 0 \end{bmatrix} P \begin{bmatrix} 0 & \delta \\ 0 & \delta \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \delta \\ 1 & 0 \\ 0 & \delta \end{bmatrix} P \begin{bmatrix} 0 & \delta \\ 0 & \delta \\ 0 & \delta \end{bmatrix}.$$

By anonymity,

$$\begin{bmatrix} 0 & \delta \\ 1 & 0 \\ 1 & 0 \end{bmatrix} I \begin{bmatrix} 1 & 0 \\ 0 & \delta \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \delta \\ 1 & 0 \\ 0 & \delta \end{bmatrix} I \begin{bmatrix} 0 & \delta \\ 0 & \delta \\ 1 & 0 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} P \begin{bmatrix} 1 & 0 \\ 0 & \delta \\ 1 & 0 \end{bmatrix} I \begin{bmatrix} 0 & \delta \\ 1 & 0 \\ 1 & 0 \end{bmatrix} P \begin{bmatrix} 0 & \delta \\ 0 & \delta \\ 1 & 0 \end{bmatrix} I \begin{bmatrix} 0 & \delta \\ 1 & 0 \\ 0 & \delta \end{bmatrix} P \begin{bmatrix} 0 & \delta \\ 0 & \delta \\ 0 & \delta \end{bmatrix},$$

and transitivity implies

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} P \begin{bmatrix} 0 & \delta \\ 0 & \delta \\ 0 & \delta \end{bmatrix}.$$

This is a contradiction. ■

As can be seen from the proof of Theorem 5, the result is true even on the restricted domains $\mathcal{D}_L$ and $\mathcal{D}_H$. Let us consider the following redistribution principles, which are weaker than the ex-post equalization principle.

Ex-post equalization principle below the threshold: For all $u \in \mathcal{D}^e \cap \mathcal{D}_L$ and for all $v \in (\mathcal{D} \cap \mathcal{D}_L) \setminus \mathcal{D}^e$, if $\sum_{i \in N} u_i^s = \sum_{i \in N} v_i^s$ for all $s \in S$, then uPv.

Ex-post equalization principle above the threshold: For all $u \in \mathcal{D}^e \cap \mathcal{D}_H$ and for all $v \in (\mathcal{D} \cap \mathcal{D}_H) \setminus \mathcal{D}^e$, if $\sum_{i \in N} u_i^s = \sum_{i \in N} v_i^s$ for all $s \in S$, then uPv.

Using the restricted versions of weak Pareto for equal risk, continuity, and the ex-post equalization principle, we obtain the following corollary. It shows that weak Pareto for equal risk is incompatible with any ex-post sufficientarian ordering associated with a transformation g that is strictly concave on $\mathcal{D}_L$ or on $\mathcal{D}_H$.

Corollary 2. There exists no social ordering R that satisfies weak Pareto for equal risk below (above) the threshold, continuity below (above) the threshold, anonymity, prospect independence of the unconcerned, and the ex-post equalization principle below (above) the threshold.

The incompatibility between weak Pareto for equal risk and the entire class of ex-post sufficientarian orderings can be explained by a fundamental tension between ex-ante efficiency and an ex-post sufficientarian approach to the evaluation of prospects. As noted earlier, the view that absolute priority should be given to people below the threshold constitutes the core of sufficientarianism. The axiom of ex-post absolute priority is a natural formalization of this idea in the context of evaluating prospects. As the following theorem shows, ex-post absolute priority and weak Pareto for equal risk are incompatible.

Theorem 6. There exists no social ordering R that satisfies weak Pareto for equal risk and ex-post absolute priority.

Proof. Let $a, b \in \mathbb{R}$ be such that $a < \theta < b$ and

$$\pi^1 a + \sum_{s \in \{2,\ldots,m\}} \pi^s b > \theta.$$

Consider the distributions $u, v \in \mathcal{D}^e$ defined by

$$u_i = (a, b\mathbf{1}_{m-1}) \ \text{ and } \ v_i = \theta \mathbf{1}_m$$

for all $i \in N$. Letting $w = u_L$, w is given by

$$w_i = (a, \theta \mathbf{1}_{m-1})$$

for all $i \in N$. Note that $w \in \mathcal{D}^e$ and $v \in \mathcal{D}_L$. Thus, it follows from weak Pareto for equal risk that vPw because

$$E(v_i) = \theta > \pi^1 a + \sum_{s \in \{2,\ldots,m\}} \pi^s \theta = E(w_i)$$

for all $i \in N$. Ex-post absolute priority implies that vPu. On the other hand, weak Pareto for equal risk implies that uPv since

$$E(u_i) = \pi^1 a + \sum_{s \in \{2,\ldots,m\}} \pi^s b > \theta = E(v_i)$$

for all $i \in N$. This is a contradiction. ■

In view of Theorem 6, any principle of ex-post sufficientarianism needs to abandon weak Pareto for equal risk, as long as it satisfies ex-post absolute priority. Indeed, from Corollary 1, this applies to all ex-post sufficientarian orderings.

The two impossibility results established in Theorems 5 and 6 are related to Theorem 1 of Fleurbaey (2010), who proposes an alternative approach to ex-post prioritarianism. Consider an increasing and concave function $h \colon \mathbb{R} \to \mathbb{R}$, and define the function $\Xi_h^n \colon \mathbb{R}^n \to \mathbb{R}$ by letting

$$\Xi_h^n(x) = h^{-1}\left(\frac{1}{n}\sum_{i \in N} h(x_i)\right)$$

for all $x \in \mathbb{R}^n$. The number $\Xi_h^n(u^s)$ is called the equally-distributed-equivalent (EDE) utility for u^s, provided that the ex-post evaluation of each state $s \in S$ is performed by the prioritarian evaluation $\sum_{i \in N} h(u_i^s)$. A social ordering is expected EDE prioritarian if and only if there exists an increasing and strictly concave function $h \colon \mathbb{R} \to \mathbb{R}$ such that, for all $u, v \in \mathcal{D}$,

$$uRv \ \Leftrightarrow \ \sum_{s \in S} \pi^s \Xi_h^n(u^s) \geq \sum_{s \in S} \pi^s \Xi_h^n(v^s). \tag{10}$$

See Fleurbaey (2010) for a more general definition of an expected EDE criterion. According to his result, a social ordering satisfies statewise dominance, weak Pareto for no risk, weak Pareto for equal risk, and continuity if and only if it belongs to the general class of expected EDE criteria. We note that the expected EDE criterion coincides with the utilitarian ordering whenever EDE utility is equal to average utility for each state. The utilitarian ordering in turn is consistent with prospect independence of the unconcerned—indeed, the utilitarian ordering is the only expected EDE criterion that satisfies prospect independence of the unconcerned; this is a corollary of Proposition 1 of Fleurbaey and Zuber (2013). From Theorem 5, if an expected EDE criterion satisfies the ex-post equalization principle, it is incompatible with prospect independence of the unconcerned. Furthermore, from Theorem 6, the expected EDE criteria are incompatible with ex-post absolute priority. This means that there exists a fundamental tension between the ex-post sufficientarian approach and the expected EDE approach to assessing prospects.

7 Interchangeability for equally probable states

To summarize, both ex-post prioritarianism and ex-post sufficientarianism satisfy prospect independence of the unconcerned but fail to satisfy weak Pareto for equal risk. On the other hand, expected EDE prioritarianism as defined in (10) does not satisfy prospect independence of the unconcerned but satisfies weak Pareto for equal risk. Thus, an advantage and disadvantage of each of these three social orderings boils down to the question of which of the two axioms, prospect independence of the unconcerned and weak Pareto for equal risk, they satisfy. As alluded to earlier, prospect independence of the unconcerned is a plausible separability property. But, at the same time, weak Pareto for equal risk can also be viewed as a desirable requirement.

While expected EDE prioritarian orderings satisfy an ex-ante Paretian requirement in a restricted form of weak Pareto for equal risk, this does, of course, not mean that alternative ex-ante Paretian requirements can be accommodated as well. Indeed, there are numerous ex-ante Pareto requirements that are violated by these orderings. As pointed out by Adler (2019), the expected EDE prioritarian orderings are incompatible with ex-ante Pareto in what is called a heartland case, where some are equal and others unaffected.

Although ex-post prioritarianism and ex-post sufficientarianism cannot comply with weak Pareto for equal risk, they are nevertheless capable of respecting the individuals' ex-ante evaluations in a different way. To illustrate this observation, consider a prospect u such that both Ann and Bob obtain utility levels of 100 in state 1 and zero in state 2; see Table 1. Now consider a different prospect v such that Bob gets zero in state 1 and 100 in state 2, all other things being equal. That is, Bob's ex-post utility levels are interchanged between the two states. Assuming that the two states are equally probable, the following axiom states that this change does not affect the relative goodness of these prospects. That is, the original prospect is indifferent to the prospect that is generated by this interchange.

Interchangeability for equally probable states: Suppose that there exist $s, s' \in S$ with $\pi_s = \pi_{s'}$. For all $u, v \in \mathcal{D}$, if there exist $i \in N$ such that

$$u_i^s = v_i^{s'}, v_i^s = u_i^{s'},\ v_j^{s'} = u_j^s \text{ for all } j \neq i, \text{ and } u^t = v^t \text{ for all } t \in S \setminus \{s, s'\},$$

then uIv.

This axiom can be seen as a restricted version of ex-ante Pareto indifference; note that the interchange in question does not affect anyone's claims or interests. For the individual whose

Table 1: Interchangeable prospects

Prospect u		
	State 1 (0.5)	State 2 (0.5)
Ann	100	0
Bob	100	0

Prospect v		
	State 1 (0.5)	State 2 (0.5)
Ann	100	0
Bob	0	100

utility levels are interchanged, in which state he or she receives the higher utility level 100 is just a matter of labeling states and, thus, is irrelevant to his or her ex-ante evaluation of the two prospects. For the other individuals, nothing changes. Consequently, this interchange does not affect anyone's ex-ante utilities, and $E(u_k) = E(v_k)$ holds for all $k \in N$.

It is easy to verify that ex-post prioritarian orderings satisfy this axiom because $\pi_s g(u_i^s) + \pi_{s'} g(u_i^{s'}) = \pi_s g(v_i^s) + \pi_{s'} g(v_i^{s'})$ holds for the prospects u and v considered in the axiom. However, expected EDE prioritarianism is incompatible with this axiom. To see this, consider the prospects u and v of Table 1. Assuming that everyone other than Ann and Bob receives $a \in \mathbb{R}$ in both states, we obtain

$$\begin{aligned}
\Xi_h^n(u^{s_1}) + \Xi_h^n(u^{s_2}) &= h^{-1}\left(\frac{2h(100) + (n-2)h(a)}{n}\right) + h^{-1}\left(\frac{2h(0) + (n-2)h(a)}{n}\right) \\
&> 2h^{-1}\left(\frac{h(100) + h(0) + (n-2)h(a)}{n}\right) \\
&= \Xi_h^n(v^{s_1}) + \Xi_h^n(v^{s_2})
\end{aligned}$$

because of the strict convexity of h^{-1}. This means that expected EDE prioritarianism concludes that u is better than v. Therefore, expected EDE prioritarianism cannot be neutral to the interchange that does not affect anyone's ex-ante utility and, in this sense, it cannot respect the individuals' ex-ante utilities. Consequently, the advantage of expected EDE prioritarianism in respecting individuals' ex-ante utilities is not as strong as it may appear to be.

Ex-post sufficientarianism also satisfies interchangeability for equally probable states. Indeed, if the sufficientarian threshold θ is such that $\theta \geq \max\{u_i^s, u_i^{s'}\}$ or $\theta \leq \min\{u_i^s, u_i^{s'}\}$, the argument employed to show that ex-post prioritarianism satisfies the axiom applies. Furthermore, the same argument works if $u_i^s > \theta > u_i^{s'}$. These observations follow because the censored prospects $u_L = \tilde{u}$ and $v_L = \tilde{v}$ satisfy

$$\tilde{u}_i^s = \theta = \tilde{v}_i^{s'}, \tilde{v}_i^s = \tilde{u}_i^{s'},\ \tilde{v}_j^{s'} = \tilde{u}_j^s \text{ for all } j \neq i, \text{ and } \tilde{u}^t = \tilde{v}^t \text{ for all } t \in S \setminus \{s, s'\},$$

and u_H and v_H satisfy an analogous property. Although ex-post prioritarianism and ex-post sufficientarianism do not satisfy weak Pareto for equal risk, they respect individuals' ex-ante utilities in a way that expected EDE prioritarianism does not.

8 Concluding remarks

In this paper, we employ a unified method to characterize ex-post welfare criteria over state-contingent alternatives. Our key axiom is prospect independence of the unconcerned, which is a risk-dependent variant of a well-established separability property. Adding a set of standard requirements leads to a characterization of ex-post prioritarianism. Utilizing this axiomatization, we characterize ex-post sufficientarianism. In the latter result, the axiom of ex-post absolute priority appears in addition to prospect independence of the unconcerned.

There are several tasks that remain to be addressed in future work. We focus on the case where the population is fixed but, evidently, there is considerable uncertainty regarding the size and the composition of future populations. In many countries, it is an urgent problem to address uncertainty related to well-being and population through public policies. Extending our framework to a variable-population setting may yield an important analytical tool to deal with population issues. The independence axioms introduced by Blackorby and Donaldson (1984) and by Blackorby, Bossert, and Donaldson (2005) play a significant role in population ethics under certainty. The extension of these axioms to prospects may constitute a promising path towards the examination of variable-population extensions of ex-post prioritarianism and ex-post sufficientarianism.

Appendix

Independence of Axioms in Theorem 1

Consider an increasing and continuous function $g\colon \mathbb{R} \to \mathbb{R}$. Define the ordering R^1 as follows. For all $u, v \in \mathcal{D}$, uRv if and only if

$$\sum_{s\in S} \pi^s \sum_{i\in N} g(u_i^s) \leq \sum_{s\in S} \pi^s \sum_{i\in N} g(v_i^s).$$

This ordering satisfies all axioms other than strong Pareto for no risk.

Define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} -e^{-x} \text{ if } x < 0; \\ 1 - e^{-x} \text{ if } x \geq 0. \end{cases}$$

This is an increasing function that is discontinuous at zero. Furthermore, g is concave on $(-\infty, 0)$ and on $[0, \infty)$. Define the ordering R^2 as follows. For all $u, v \in \mathcal{D}$, uR^2v if and only if

$$\sum_{s\in S} \pi^s \sum_{i\in N} g(u_i^s) \geq \sum_{s\in S} \pi^s \sum_{i\in N} g(v_i^s).$$

This ordering satisfies all axioms other than continuity.

Consider n continuous and increasing functions $g_i\colon \mathbb{R} \to \mathbb{R}$ for all $i \in N$ with the property that there exist $j, k \in N$ such that g_k is not an affine transformation of g_j. Define the ordering R^3 as follows. For all $u, v \in \mathcal{D}$, uR^3v if and only if

$$\sum_{s\in S} \pi^s \sum_{i\in N} g_i(u_i^s) \geq \sum_{s\in S} \pi^s \sum_{i\in N} g_i(v_i^s).$$

This ordering satisfies all axioms other than anonymity.

An ordering that is prioritarian with probability weighing such that the function ϕ in its definition is not linear satisfies all axioms other than the social expected-utility hypothesis.

Define the function $W^*\colon \mathbb{R}^n \to \mathbb{R}$ by

$$W^*(x) = \min\{x_1, \ldots, x_n\} + \sum_{i=1}^{n} x_i$$

for all $x \in \mathbb{R}^n$. Now define the ordering R^4 as follows. For all $u, v \in \mathcal{D}$, uR^4v if and only if

$$\sum_{s \in S} \pi^s W^*(u^s) \geq \sum_{s \in S} \pi^s W^*(v^s).$$

This ordering satisfies all axioms other than prospect independence of the unconcerned.

Independence of Axioms in Theorem 4

Let $g\colon \mathbb{R} \to \mathbb{R}$ be an increasing and continuous function. Define R^5 by letting, for all $u, v \in \mathcal{D}$, uR^5v if and only if

$$\sum_{s \in S} \pi^s \sum_{i \in L(u^s)} (g(u_i^s) - g(\theta)) < \sum_{s \in S} \pi^s \sum_{i \in L(v^s)} (g(v_i^s) - g(\theta))$$

or

$$\begin{aligned} &\sum_{s \in S} \pi^s \sum_{i \in L(u^s)} (g(u_i^s) - g(\theta)) = \sum_{s \in S} \pi^s \sum_{i \in L(v^s)} (g(v_i^s) - g(\theta)) \text{ and} \\ &\sum_{s \in S} \pi^s \sum_{i \in H(u^s)} (g(u_i^s) - g(\theta)) \leq \sum_{s \in S} \pi^s \sum_{i \in H(v^s)} (g(v_i^s) - g(\theta)). \end{aligned}$$

This ordering satisfies all axioms other than strong Pareto for no risk.

Let $g\colon \mathbb{R} \to \mathbb{R}$ be an increasing function that is not continuous at a point below θ. Define R^6 by letting, for all $u, v \in \mathcal{D}$, uR^6v if and only if

$$\sum_{s \in S} \pi^s \sum_{i \in L(u^s)} (g(u_i^s) - g(\theta)) > \sum_{s \in S} \pi^s \sum_{i \in L(v^s)} (g(v_i^s) - g(\theta))$$

or

$$\begin{aligned} &\sum_{s \in S} \pi^s \sum_{i \in L(u^s)} (g(u_i^s) - g(\theta)) = \sum_{s \in S} \pi^s \sum_{i \in L(v^s)} (g(v_i^s) - g(\theta)) \text{ and} \\ &\sum_{s \in S} \pi^s \sum_{i \in H(u^s)} (g(u_i^s) - g(\theta)) \geq \sum_{s \in S} \pi^s \sum_{i \in H(v^s)} (g(v_i^s) - g(\theta)). \end{aligned}$$

This ordering satisfies all axioms other than continuity below the threshold.

Let $g\colon \mathbb{R} \to \mathbb{R}$ be an increasing function which is not continuous at a point above θ. Define R^7 in analogy to R^6. Clearly, R^7 satisfies all axioms other than continuity above the threshold.

Consider n continuous and increasing functions $g_i\colon \mathbb{R} \to \mathbb{R}$ for all $i \in N$ with the property that there exist $j, k \in N$ such that g_k is not an affine transformation of g_j. Define the ordering R^8 by letting, for all $u, v \in \mathcal{D}$, uR^8v if and only if

$$\sum_{s\in S} \pi^s \sum_{i\in L(u^s)} (g_i(u_i^s) - g_i(\theta)) > \sum_{s\in S} \pi^s \sum_{i\in L(v^s)} (g_i(v_i^s) - g_i(\theta))$$

or

$$\sum_{s\in S} \pi^s \sum_{i\in L(u^s)} (g_i(u_i^s) - g_i(\theta)) = \sum_{s\in S} \pi^s \sum_{i\in L(v^s)} (g_i(v_i^s) - g_i(\theta)) \text{ and}$$
$$\sum_{s\in S} \pi^s \sum_{i\in H(u^s)} (g_i(u_i^s) - g_i(\theta)) \geq \sum_{s\in S} \pi^s \sum_{i\in H(v^s)} (g_i(v_i^s) - g(\theta)).$$

This ordering satisfies all axioms other than anonymity.

An ordering that is prioritarian with probability weighing such that the function ϕ in its definition is not linear satisfies all axioms other than the expected sufficientarian hypothesis.

Define R^9 by letting, for all $u, v \in \mathcal{D}$,

$$\begin{aligned} uR^9v \;\Leftrightarrow\; & \sum_{s\in S} \pi^s \Xi_h^n(u_L^s) > \sum_{s\in S} \pi^s \Xi_h^n(v_L^s) \;\text{ or} \\ & \left[\sum_{s\in S} \pi^s \Xi_h^n(u_L^s) = \sum_{s\in S} \pi^s \Xi_h^n(v_L^s) \;\text{ and }\; \sum_{s\in S} \pi^s \Xi_h^n(u_H^s) \geq \sum_{s\in S} \pi^s \Xi_h^n(v_H^s)\right]. \end{aligned}$$

This ordering satisfies all axioms other than prospect independence of the unconcerned.

Independence of the utilities of the sure

The independence axiom used by Fleurbaey and Zuber (2013) is formally stated as follows.

Independence of the utilities of the sure: For all $u, v \in \mathcal{D}$, for all $u', v' \in \mathcal{D}^c$, and for all $\emptyset \neq M \subsetneq N$,

$$(u_M, v_{N\setminus M})R(u'_M, v_{N\setminus M}) \Leftrightarrow (u_M, v'_{N\setminus M})R(u'_M, v'_{N\setminus M}).$$

According to this axiom, social evaluations are not affected by unconcerned individuals whose utility levels are constant across states. It is obvious that independence of the utilities of the sure is logically weaker than prospect independence of the unconcerned.

One might ask if this weaker axiom is sufficient to establish our characterization of ex-post generalized utilitarianism (or sufficientarianism). The answer is no.

Let $g\colon \mathbb{R} \to \mathbb{R}$ be an increasing and continuous function, and define the function $\Lambda_g\colon \mathbb{R}^n \to \mathbb{R}$ by letting

$$\Lambda_g(x) = K^{\sum_{i\in N} g(x_i)}$$

for all $x \in \mathbb{R}^n$, where K is a constant larger than one. Define R by letting, for all $u, v \in \mathcal{D}$,

$$uRv \Leftrightarrow \sum_{s\in S} \pi_s \Lambda_g(u^s) \geq \sum_{s\in S} \pi_s \Lambda_g(v^s).$$

The ordering R satisfies strong Pareto for no risk, continuity, anonymity, the social expected-utility hypothesis, and independence of the utilities of the sure. However, prospect independence of the unconcerned is not satisfied. A similar example can be used to show that independence of the utilities of the sure is not sufficient to characterize ex-post sufficientarianism.

Which independence axiom to impose on the social ordering R—prospect independence of the unconcerned or the weaker axiom, independence of the utilities of the sure—is a normative question. Addressing that question in detail lies beyond the scope of this Article, but we believe that a good case can be made for the stronger axiom. The choice of axiom depends upon the interpretation of prospects and states. In decision theory, a mapping from states to outcomes represents a possible choice (action) for a decisionmaker in some choice situation. Probabilities assigned to states encode the decisionmaker's uncertainty. "Each state ... is a compilation of all characteristics/factors about which [the decisionmaker] is uncertain and which are relevant to the consequences that will ensue from his choice." (Kreps 1988, p. 34). A prospect, on the decision-theoretic interpretation, represents a possible social choice, mapping each state onto a vector of utilities for everyone in the population. The choices available to some social decisionmaker ("social planner") at a point in time are represented by the corresponding set of prospects.

The idea behind prospect independence of the unconcerned is that the comparison of two possible social choices, represented by prospects u and v, should be independent of the utility of anyone who is sure to be unaffected by the choice: the choice is sure not to affect the person's utility, because in every state of nature the person's utility is the same with u as with v. But note that the social planner may not know for certain what the utility level of a sure-to-be-unaffected person is. Consider individuals who are already dead at the time of choice. Although the social planner can be sure that the utility of the dead will be unaffected by her decision, she may well not know what their well-being levels were—since the past is one of the things she may be uncertain about. The dead satisfy prospect independence of the unconcerned, but may well not satisfy independence of the utility of the sure.

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