Sharing the revenues from broadcasting sport events*

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Abstract

We study the problem of sharing the revenue from broadcasting sport events, among participating players. We provide direct, axiomatic and game-theoretical foundations for two focal rules: the *Shapley* rule and the *concede-and-divide* rule. The former allocates the revenues from each game equally among the participating players. The latter assigns to each player the revenue from the differential audience with respect to the average audience per game that the rest of the players yield (in the remaining games they play). Both rules are polar in their treatment of the *fan effect*. We also provide an application studying the case of sharing the revenue from broadcasting games in *La Liga*, the Spanish football League. Our rules indicate that, somewhat surprisingly, the sharing schemes implemented by La Liga are biased against the teams driving larger audiences.

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1 Introduction

For most sports organizations, the sale of broadcasting and media rights is now the biggest source of revenue. A study of how much money various professional sports leagues generates shows that the NFL made \$13 billion in revenue last season.[1] The Major League Baseball, came second with \$9.5 billion and the Premier League third with \$5.3 billion.[2] Sharing these sizable revenues among participating teams is, by no means, a straightforward problem. Rules vary across the world. For instance, FC Barcelona and Real Madrid CF, the two Spanish giant football clubs, used to earn each more than 20% of the revenues generated by the Spanish football league. In England, however, the top two teams combined only make 13% of the revenues generated by the Premier League.[3]

The aim of this paper is to provide a formal model to study the problem of sharing the revenues from broadcasting sport events. Our model could be applied to different forms of competitions, but our running example will be the format of most European football leagues. That is, a round robin tournament in which each competitor (team) plays in turn against every other (home and away). Thus, the input of our model will be a (square) matrix in which each entry will be indicating the revenues associated to broadcasting the game between the two corresponding competitors. For ease of exposition, we shall assume an equal *pay per view* fee to each game. Thus, broadcasting revenues can be normalized to audiences.

We shall take several approaches to analyze this problem. Two rules will be salient from our analysis. On the one hand, what we shall call the *Shapley* rule, which allocates the revenues from each game equally among the two playing teams, and aggregates across games. On the other hand, what we shall call the *concede-and-divide* rule, which assigns to each team the revenue from the differential audience with respect to the average audience per game that the rest of the teams yield (in the remaining games they play). As we shall explain later, both rules convey somewhat polar forms of estimating the *fan effect*.

[1] The study "Which Professional Sports Leagues Make the Most Money" is published by Howmuch.net, a cost information website. It can be accessed at https://howmuch.net/articles/sports-leagues-by-revenue.

[2] Four of the top five leagues in revenue are in North America. However, 14 of the 20 biggest earners are football leagues that are mostly based in Europe.

[3] This might partly explain why in the last 13 editions of the Spanish football league only 1 time the champion was different from FC Barcelona and Real Madrid CF, whereas the Premier League witnessed 4 different champions in its last 5 editions.

More precisely, we first take a direct approach and suppose that the audience of each game involving two teams is divided in four (disjoint) groups; namely, the generic fans of the sport being played (who watch the game, independently of the involved teams), the fans of each team (who watch the game, independently of the opponent), and the joint fans of both teams (who watch the game, because those two teams play). We then consider two focal scenarios for what we call the *fan effect*. The *pessimistic* scenario assumes that no team has independent fans and that, therefore, the audience of each game should be equally attributed to both teams. This gives rise to the *Shapley* rule. The *optimistic* scenario assumes that no joint fans exist and, therefore, each team should be attributed the audience associated to its fan base. If generic fans of the sport being played also exist, they should be split equally, among all teams. In other words, each team concedes the other the audience attributed to its fan base, and the remainder audience is divided equally. This gives rise to the *concede-and-divide* rule.

In the *pessimistic* scenario for the fan effect described above, we also take a strategic approach in which we deal with a natural cooperative game associated to the problem. It turns out that the Shapley value of such a game will always yield the same solutions as the *Shapley* rule for the original problem (hence its name). Due to the properties of the game, the Shapley value also coincides with two other well-known values (the Nucleolus and the τ-value), and it is guaranteed to be a selection of the *core*, which implies that the *Shapley* rule guarantees the standard participation constraints. In other words, the allocations provided by the *Shapley* rule are *secession-proof*, as they do not provide teams with incentives to secede from the initial organization and create their own (sub)tournament.

In this (*pessimistic*) scenario we also take another indirect approach in which we focus on an associated problem of adjudicating conflicting claims to the original problem. Here we observe that two of the most well-known rules to adjudicate conflicting claims (the proportional and Talmud rules), which can be traced back to Aristotle and the Talmud (e.g., Moreno-Ternero and Thomson, 2017) coincide with our *Shapley* rule in their recommendations. The other two most well-known rules to adjudicate conflicting claims (the constrained equal-awards and constrained equal-losses rules) do not always guarantee *secession-proof* allocations.

As for the *optimistic* scenario for the fan effect, we derive the *concede-and-divide* rule as an intuitive procedure of sharing audiences, partly based on a linear regression model. More precisely, if we assume the audience of a game is disentangled in four numbers, referring to the four groups mentioned above, and aim to minimize the fourth (referring to the joint fans of both teams), then the problem would be equivalent to deriving the OLS estimator of a suitable linear regression model (after dealing with an eventual problem of colinearity). We show that if we compute the OLS estimations for each of the four numbers in which an audience disentangles, the rule constructed imposing the principle of concede-and-divide to those estimations happens to coincide with the *concede-and-divide* rule.

Finally, we also take an axiomatic approach to the problem formalizing axioms that reflect ethical or operational principles with normative appeal. It turns out that the two rules mentioned above are characterized by three properties. Two properties are common in both characterizations. Namely, *equal treatment of equals*, which states that if two competitors have the same audiences, then they should receive the same amount, and *additivity*, which states that revenues should be additive on the audience matrix.[4] The third property in each characterization result comes from a pair of somewhat polar properties modeling the effect of *null* or *nullifying* players, respectively. More precisely, the *null player* property says that if nobody watches a single game of a given team (i.e., the team has a null audience), then such a team gets no revenue. On the other hand, the *nullifying player* property says that if a team nullifies the audience of all the games it plays (for instance, due to some kind of boycott), then the allocation of such a team should decrease exactly by the total audience of such a team.[5]

We conclude our analysis with an empirical application focussing on La Liga, the Spanish Football League, a tournament fitting our model. We provide the schemes that our two rules would yield for the available data from that tournament. They convey somewhat controversial insights. To begin with, both schemes suggest that, somewhat surprisingly, the current existing schemes are biased against the three teams driving the largest audiences. Furthermore, the scheme provided by the *concede-and-divide* rule suggests that some teams in the tournament should compensate the others (rather than receiving broadcasting revenues) for being part of it. Partly due to these features, we also explore hybrid schemes that would alleviate them.

[4]An interpretation is that the aggregation of the revenue sharing in two seasons (involving the same competitors) is equivalent to the revenue sharing in the hypothetical combined season aggregating the audiences of the corresponding games (involving the same teams) in both seasons.

[5]It turns out that, as we shall show later, additivity is implied by the *nullifying player* property, when combined with *equal treatment of equals*, and, thus, it is not required in the characterization of the *concede-and-divide* rule.

Our work is related to several research fields, as described next.

First, it is obviously connected to the literature on sport economics. In his pioneering work within that literature, Rottenberg (1956) argued that, under the profit-maximizing assumption, revenue sharing among (sport) clubs does not affect the distribution of playing talent.[6] This was later contested in more general models (e.g., Atkinson et al., 1988; Késenne, 2000). In any case, the distribution of playing talent determines the competitive balance of a sports competition and, therefore, its value (e.g., Hansen and Tvede, 2016). We are not concerned in this paper with the process of transforming revenues into playing talent that each team undertakes. Our aim, instead, is to explore appealing rules (from a normative, as well as from a strategic viewpoint) to share the revenues that are collectively obtained upon selling broadcasting rights. To the best of our knowledge, this has not been addressed in the literature on sport economics yet.

Second, our work also relates to the industrial organization literature dealing with bundling. It has long been known that bundling products may increase revenue with respect to selling products independently (e.g., Adams and Yellen, 1976). Industries traditionally engaged in the practice include telecommunications, financial services, health care, and information. Transportation cards combining access to all the transportation means (e.g., bus, subway, tram) in a given city, or cultural cards doing the same for cultural venues (e.g., museums, attractions) are also frequent cases (e.g., Bergantiños and Moreno-Ternero, 2015). In the hyper-connected world we live in nowadays, within the era of internet, new bundling strategies are emerging. Focal instances are unlimited streaming video or music downloads through periodic charges from digital video merchants or music sellers. There exist complex relationships between the independent price (pay per view/listening) of each product and the bundled price. Consequently, the problem of sharing the revenue from periodic charges to unlimited streaming among the participating agents (authors, artists, etc.) is a complex one. Nevertheless, it shares many features with the problem we analyze in this paper. Thus, we believe our results could shed light on analyzing that problem too.

Third, our work is also connected to the axiomatic literature on resource allocation. In the last forty years, a variety of formal criteria of fair allocation have been introduced in economic theory (e.g., Thomson, 2014). These criteria have broad conceptual appeal, as well as significant

[6]See also El Hodiri and Quirk (1971).

operational power, and have contributed considerably to our understanding of normative issues concerning the allocation of goods and services. The pioneering (and successful) criterion was no-envy (e.g., Foley, 1967), which simply says that no agent should prefer someone else's assignment to his own. Other criteria formalizing ethical principles such as impartiality, priority, or solidarity have also played an important role in deriving fair allocation rules (e.g., Moreno-Ternero and Roemer, 2006, 2012).

Fourth, our paper is also related to the literature on cooperative game theory. There has been some tradition in analyzing problems involving agents' cooperation with a game-theoretical approach. Classical instances are the so-called airport problems (e.g., Littlechild and Owen, 1973), in which the runway cost has to be shared among different types of airplanes with a linear graph representing the runway, bankruptcy problems from the Talmud (Aumann and Maschler, 1985), or telecommunications problems such as the Terrestrial Flight Telephone System (in short, TFTS) and the rerouting of international telephone calls (e.g., van den Nouweland et al., 1996). One of the approaches we take in this paper is precisely this one. The game we associate to our problems is formally equivalent to the game associated by van den Nouweland et al., (1996) to the TFTS situations they study. This implies that several traditional values (Shapley, Nucleolus and τ) coincide for the game, and that they represent core selections, thus guaranteeing that the participation constraints are satisfied. As we shall argue later, this will constitute a strong argument in favor of the Shapley rule (which coincides with the mentioned three values in this case).

To conclude, we mention that, in a certain sense, we can interpret a game between two teams as a joint venture with which they generate revenues. As such, the *fan effect* could be interpreted as a measure of each team's marginal productivity for this joint venture. Flores-Szwagrzak and Treibich (2016) have recently introduced an innovative productivity index, dubbed *CoScore*, that disentangles individual from collaborative productivity. Although they apply it to formally account for coauthorship in quantifying individual scientific productivity, it is potentially valid for our setting too. Formally, they consider an academic database C and a group of authors N, which, in our setting, would be the games played in a tournament, and the group of teams. Each paper p is described by its group of coauthors $S(p)$ and a cardinal measure of scientific worth $w(p)$. In our setting, for each game $p \in C$, $w(p) = a_{ij} + a_{ji}$, where a_{ij} denotes the game teams i and j play at i's stadium, and $S(p) = \{i, j\}$. For any such database, their *CoScore*

$s \in \mathbb{R}_+^N$ is such that, for each author (team) i,

$$s_i = \frac{1}{a_i} \sum_{p \in C_i} w(p) \frac{s_i^\alpha}{\sum_{j \in S(p)} s_j^\alpha},$$

where $\alpha \in [0,1]$ is a fixed parameter that determines how much credit should be allocated to more productive authors (teams).[7] It turns out that, if there exists a vector of fans $\{b_i\}_{i \in N}$ such that $a_{ij} = b_i + b_j$ for each pair $i, j \in N$, and $\alpha = 1$, then the *CoScore* coincides precisely with the *concede-and-divide* rule in our setting.

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3, we deal with the pessimistic scenario for the fan effect, and two indirect approaches, which all drive towards the Shapley rule. More precisely, we take in this section a game-theoretical approach, associating a suitable cooperative game to each problem, which constitutes an endorsement for the Shapley rule. We also associate our problems to problems of adjudicating conflicting claims and appeal to focal rules in the sizable literature dealing with these later problems to solve the former. In Section 4, we deal with the optimistic scenario for the fan effect, which drives towards the concede-and-divide rule. In Section 5, we present the axiomatic analysis. In Section 6, we provide an empirical application on the Spanish Football League. Finally, we conclude in Section 7.

2 The model

Let $\mathbb{N}$ represent the set of all potential competitors (**teams**) and let $\mathcal{N}$ be the family of all finite (non-empty) subsets of $\mathbb{N}$. An element $N \in \mathcal{N}$ describes a finite set of teams. Its cardinality is denoted by n. We assume $n \geq 3$.[8] Given $N \in \mathcal{N}$, let Π_N denote the set of all orders in N. Given $\pi \in \Pi_N$, let $Pre\,(i, \pi)$ denote the set of elements of N which come before i in the order given by π, *i.e.*, $Pre\,(i, \pi) = \{j \in N \mid \pi\,(j) < \pi\,(i)\}$. For notational simplicity, given $\pi \in \Pi_N$, we denote the agent $i \in N$ with $\pi\,(i) = s$ as π_s.

For each pair of teams $i, j \in N$, we denote by a_{ij} the broadcasting audience (number of viewers) for the game played by i and j at i's stadium. We use the notational convention that

[7] Flores-Szwagrzak and Treibich (2016) show that the CoScore rule is uniquely characterized by three axioms: consistency, invariance to merging papers, and invariance to merging coauthors.

[8] All of our results hold under this assumption, and some of them hold too when $n = 2$.

$a_{ii}=0$, for each $i \in N$. Let $A=(a_{ij})_{(i,j)\in N\times N}$ denote the resulting matrix with the broadcasting audiences generated in the whole tournament involving the teams within N.[9] Let $\mathcal{A}_{n\times n}$ denote the set of all possible such matrices (with zero entries in the diagonal), and $\mathcal{A}=\bigcup_n \mathcal{A}_{n\times n}$.

For each $A \in \mathcal{A}$, let $||A||$ denote the total audience of the tournament. Namely,

$$||A|| = \sum_{i,j\in N} a_{ij}.$$

A (broadcasting sports) **problem** is a duplet (N,A), where $N \in \mathcal{N}$ is the set of **teams** and $A=(a_{ij})_{(i,j)\in N\times N} \in \mathcal{A}_{n\times n}$ is the **audience matrix**. The family of all the problems described as such is denoted by $\mathcal{P}$. When no confusion arises we write A instead of (N,A).

For each $(N,A)\in\mathcal{P}$, and each $i\in N$, let $\alpha_i(A)$ denote the total audience achieved by team i, i.e.,

$$\alpha_i(A) = \sum_{j\in N}(a_{ij}+a_{ji}).$$

When no confusion arises we write α_i instead of $\alpha_i(A)$. Notice that, for each problem $(N,A)\in\mathcal{P}$, $\sum_{i\in N}\alpha_i(A) = 2||A||$.

Consider the following example, which will be used often in the paper.

Example 1 *Let* $(N,A)\in\mathcal{P}$ *be such that* $N=\{1,2,3\}$ *and*

$$A = \begin{pmatrix} 0 & 1200 & 1030 \\ 1200 & 0 & 230 \\ 1030 & 230 & 0 \end{pmatrix}$$

Then, $||A|| = 4920$ *and* $\alpha(A) = (\alpha_1(A),\alpha_2(A),\alpha_3(A)) = (4460, 2860, 2520)$.

A (sharing) **rule** is a mapping that associates with each problem an allocation indicating the amount each team gets from the total revenue generated by broadcasting games. Without

[9] We are therefore assuming a standard round robin tournament, i.e., a league in which each team plays each other team twice: once home, another away. This is the usual format, for instance, of the main European football leagues. Our model could also be extended to encompass other formats such as those in which some teams play other teams a different number of times, or even include play-offs at the end of the regular season, which is the case of most of North American Professional Sports.

loss of generality, we normalize the revenue generated by each viewer to 1 (to be interpreted as the "pay per view" fee). Thus, formally, $R : \mathcal{P} \rightarrow \mathbb{R}^n$ is such that, for each $(N, A) \in \mathcal{P}$,

$$\sum_{i \in N} R_i (A) = ||A||.$$

Some examples of rules are:

- The **Equal Awards rule**, E, which divides the total audience equally among the teams. Namely, for each problem $(N, A) \in \mathcal{P}$, and each $i \in N$,

$$R_i (N, A) = \frac{||A||}{n}.$$

- The **Shapley rule**, S, divides the total audience proportionally to the vector of audiences of the teams.[10] Namely, for each problem $(N, A) \in \mathcal{P}$, and each $i \in N$,

$$S_i (N, A) = \frac{\alpha_i}{2} = \frac{\sum_{j \in N} (a_{ij} + a_{ji})}{2}.$$

- The **concede-and-divide rule**, CD proposes a specific linear combination of the Equal Awards rule and the Shapley rule.[11] Namely, for each $(N, A) \in \mathcal{P}$, and each $i \in N$,

$$CD_i (N, A) = \frac{(n-1)\, \alpha_i - ||A||}{n-2} = \alpha_i - \frac{\sum\limits_{j,k \in N \setminus \{i\}} (a_{jk} + a_{kj})}{n-2}.$$

  Notice that $\sum\limits_{j,k \in N \setminus \{i\}} (a_{jk} + a_{kj})$ is the total audience in the $(n-1)(n-2)$ games played by the rest of the teams. Thus, $\frac{\sum_{j,k \in N \setminus \{i\}} (a_{jk} + a_{kj})}{n-2}$ is the average audience per game in the games played by each of the rest of the teams. Thus, the rule is assigning to each team the differential audience with respect to the average audience per game that the rest of the teams yield (in the remaining games they play).

In **Example 1** we have that

Rule/Team	1	2	3
Equal Awards	1640	1640	1640
Shapley	2230	1430	1260
Concede-and-Divide	4000	800	120

[10] The reader is referred to Section 3 for a plausible reason to name this rule after Shapley (1953).

[11] The reader is referred to the end of this section for a plausible reason to name this rule as such.

Both the Equal Awards rule and the Shapley rule ignore the existence of team fans. More precisely, the former rule allocates the aggregate audience equally among all teams, whereas the latter splits the audience of each game equally among the two participating teams. Now, suppose a given game between teams i and j is watched (via broadcasting) by, say, 200 people. Given our normalizing assumption, this means the game generates a revenue of 200. Imagine now we have the following additional information: 20 individuals watched the game simply because they like the sport (and might had watched any game being broadcasted). 100 individuals watched the game because they are fans of team i, whereas 30 individuals watched the game because they are fans of team j. Finally, the remaining 50 individuals watched the game because they considered that particular game between teams i and j was appealing (at least, ex ante). The Shapley rule would propose awarding 100 to each team thus ignoring the unequal number of fans for both teams.[12] An alternative allocation, taking into account this latter aspect, would concede each team the amount generated by its fans (100 for team i, 30 for team j) and would divide equally the rest. That is, team i would receive 135 whereas team j would receive 65. As we shall see next, this *concede-and-divide* procedure, which can be traced back to the Talmud, paves the way for the rule we defined above under the same name.[13]

The *fan effect* described above is relevant in practice. It might actually explain (at least, partially) why audiences differ so much. Some teams have more fans than others and, consequently, they drive larger audiences. This aspect seems to be indeed captured by the actual revenue sharing process used in professional sports, where the amount assigned to each team depends on some parameters that try to capture such heterogeneity.

We can safely argue that, in general, one might become a viewer of a game involving teams i and j for several reasons:

1. Because of being a fan of this sport per se (in which case one would be eager to watch all the games, independently of the teams playing).

2. Because of being a fan of team i (in which case one would be eager to watch all the games involving team i).

[12] So would do the Equal Awards rule, provided the tournament is only made of this game.

[13] The name was coined by Thomson (2003) to illustrate the solution to the so-called contested-garment problem appearing in the Talmud.

3. Because of being a fan of team j (in which case one would be eager to watch all the games involving team j).

4. Because of considering that the games between teams i and j are interesting.

The problem is that, in practice, the above information is not available and we only know the total audience of the game. Let us, for instance, revisit **Example 1**. Therein, we can conjecture several plausible explanations (in terms of items 1 to 4 described above) for the provided audiences.

Explanation (a). All viewers belong to group 4 and, thus, no team has fans. In this case, the procedure described above would recommend awarding team 1 with

$$\frac{1200}{2}+\frac{1030}{2}+\frac{1200}{2}+\frac{1030}{2}=2230.$$

More generally, the allocation would be

$$(2230, 1430, 1260),$$

which coincides precisely with the allocation proposed by the Shapley rule in this example.

Explanation (b). Team 1 has 1000 fans, team 2 has 200 fans and team 3 has 30 fans. No viewers belong to groups 1 or 4. In this case, the procedure described above would recommend awarding team 1 with 4000 (it plays 4 games with 1000 fans in each). More generally, the allocation would be

$$(4000, 800, 120),$$

which coincides precisely with the allocation proposed by the concede-and-divide rule in this example.

Explanation (c). Team 1 has 800 fans, team 2 has 100 fans and team 3 has 30 fans. 90 viewers belong to group 1. The rest of the viewers belong to group 4. That is,

Totals	Group 1	Fans 1	Fans 2	Fans 3	Group 4
1200	90	800	100		210
1030	90	800		30	110
230	90		100	30	10

In this case, the procedure described above would recommend the allocation

$$(3700, 800, 420),$$

which is in between the other two allocations described above.

The previous examples can be generalized as follows:

In the first scenario, it is assumed that there are no fans. Thus, it seems natural that, for each $i, j \in N$, a_{ij} is divided equally between teams i and j. In other words, each team will receive half of the audiences of each game it plays. This is precisely what the Shapley rule recommends.

In the second scenario, we consider as many fans as possible, compatible with the data. We do so upon minimizing the number of viewers in group 4. This is the most optimistic scenario for the computation of fans. As we shall see later, this is precisely what the concede-and-divide rule recommends.

The two rules therefore provide polar estimations of the fan effect. Based on this, it seems natural to argue that they should provide a range in which allocations estimating the fan effect should lie. For instance, in Example 1, team 1 should receive something between 2230 and 4000, team 2 between 800 and 1430 and team 3 between 120 and 1260.

3 The pessimistic scenario for the fan effect and the Shapley rule

In this section we assume the most pessimistic scenario for the fan effect described above. In the parlance used above, we assume that all viewers belong to group 4 and, therefore, we assume teams have no fans. This means that a person decides to watch a game only because of the combination of teams playing the game.

In this scenario, and as argued above, it seems natural to consider the Shapley rule. We then analyze it here in detail. In the first subsection, we associate to each problem a cooperative game with transferable utility. We prove that the Shapley value of the game coincides with the Shapley rule, which explains the name of the rule. The core is non empty and the Shapley rule belongs to the core. In the second subsection, we associate to each problem a claims problem. We prove that the so-called proportional and Talmud rules for claims problems coincide with Shapley rule.

3.1 The (cooperative) game-theoretical approach

A **cooperative game with transferable utility**, briefly a **TU game**, is a pair (N, v), where N denotes a set of agents and $v : 2^N \to \mathbb{R}$ satisfies that $v(\varnothing) = 0$. We say that (N, v) is **convex** if, for each pair $S, T \subset N$ and $i \in N$ such that $S \subset T$ and $i \notin T$,

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$

Given $S \subset N$, the **unanimity game** associated with S is defined as the TU game (N, u_S) where $u_S(T) = 1$ if $S \subset T$, and $u_S(T) = 0$ otherwise. Given a TU game (N, v), there exists a unique family of numbers $\{\delta_S\}_{S \subset N}$ such that $v = \sum\limits_{S \subset N} \delta_S u_S$.

We present some well-known solutions for TU games. First, the **core**, defined as the set of feasible payoff vectors, for which no coalition can improve upon. Formally,

$$C(N, v) = \left\{ x \in \mathbb{R}^N \text{ such that } \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S), \text{ for each } S \subset N \right\}.$$

The **Shapley value** (Shapley, 1953) is the linear function that, for each unanimity game, splits each unit equally among the members of the coalition (and only among them). Formally, for each $i \in N$, $Sh_i(N, v) = \sum\limits_{S \subset N} \delta_S Sh_i(N, u_S)$, where

$$Sh_i(N, u_S) = \begin{cases} \frac{1}{|S|} & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, we can define it as follows: for each $i \in N$,

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \left[v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi)) \right].$$

We associate with each (broadcasting sports) problem $(N, A) \in \mathcal{P}$ a TU game (N, v_A) where, for each $S \subset N$, $v_A(S)$ denotes the total audience of the games played by the teams in S. Namely,

$$v_A(S) = \sum_{\substack{i,j \in S \\ i \neq j}} a_{ij} = \sum_{\substack{i,j \in S \\ i < j}} (a_{ij} + a_{ji}).$$

Notice that, for each problem $(N, A) \in \mathcal{P}$ and each $i \in N$, $v_A(\{i\}) = 0$.

In Example 1 we have that

S	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_A(S)$	2400	2060	460	4920

and

$$Sh(N, v_A) = (2230, 1430, 1260) = S(N, A).$$

The next result establishes a correspondence between the Shapley rule and the Shapley value for the TU-game described above, which justifies the name given to the rule.

Theorem 1 *For each $(N, A) \in \mathcal{P}$, $S(N, A) = Sh(N, v_A)$.*

Proof. Let $(N, A) \in \mathcal{P}$ and (N, v_A) be its associated TU game. For each pair $i, j \in N$ with $i \neq j$ we define the characteristic function v_A^{ij} as follows. For each $S \subset N$,

$$v_A^{ij}(S) = \begin{cases} a_{ij} + a_{ji} & \text{if } \{i, j\} \subset S \\ 0 & \text{otherwise.} \end{cases}$$

Consider the resulting TU-game $\left(N, v_A^{ij}\right)$. It is straightforward to see that, for such a game, agents i and j are symmetric, whereas the remaining agents in $N \backslash \{i, j\}$ are null teams. Thus,

$$Sh_k\left(N, v_A^{ij}\right) = \begin{cases} \frac{a_{ij} + a_{ji}}{2} & \text{if } k \in \{i, j\} \\ 0 & \text{otherwise.} \end{cases}$$

For each $S \subset N$,

$$v_A(S) = \sum_{\substack{i,j \in S \\ i<j}} (a_{ij} + a_{ji}) = \sum_{\substack{i,j \in N \\ i<j}} v_A^{ij}(S).$$

As the Shapley value is additive on v, we have that

$$Sh(N, v_A) = \sum_{\substack{i,j \in N \\ i<j}} Sh\left(N, v_A^{ij}\right).$$

Thus, for each $k \in N$,

$$Sh_k(N, v_A) = \sum_{\substack{i,j \in N \\ i<j}} Sh_k\left(N, v_A^{ij}\right) = \sum_{j \in N} Sh_k\left(N, v_A^{kj}\right) = \sum_{j \in N} \frac{a_{kj} + a_{jk}}{2} = \frac{\alpha_k}{2}.$$

■

The game we have described in this section is formally equivalent to the game associated by van den Nouweland et al., (1996) to the so-called Terrestial Flight Telephone System (in short, TFTS) situations they formalize.[14] They prove that such a game is convex and, therefore, its Shapley value belongs to the core. Thus, it follows from Theorem 1 that the Shapley rule always yields stable allocations, in the sense formalized by the core. Formally, $S(N,A) \in C(N,v_A)$, for each problem (N,A).

The next result fully characterizes the core of this game.

Proposition 1 *Let $(N,A) \in \mathcal{P}$ and (N,v_A) be its associated TU game. Then, $x = (x_i)_{i\in N} \in C(N,v_A)$ if and only if, for each $i \in N$, there exist $\left(x_i^j\right)_{j\in N\setminus\{i\}}$ satisfying three conditions:*

(i) $x_i^j \geq 0$, *for each* $j \in N\setminus\{i\}$;

(ii) $\sum\limits_{j\in N\setminus\{i\}} x_i^j = x_i$, *for each* $i \in N$;

(iii) $x_i^j + x_j^i = a_{ij} + a_{ji}$, *for each pair* $i,j \in N$, *with* $i<j$.

Proof. We first prove that if $x = (x_i)_{i\in N}$ is such that for each $i \in N$, there exists $\left(x_i^j\right)_{j\in N\setminus\{i\}}$ satisfying the three conditions, then $x \in C(N,v_A)$.

By (ii),

$$\sum_{i\in N} x_i = \sum_{i\in N}\sum_{j\in N\setminus\{i\}} x_i^j = \sum_{\substack{i,j\in N\\ i<j}} \left(x_i^j + x_j^i\right).$$

By (iii),

$$\sum_{\substack{i,j\in N\\ i<j}} \left(x_i^j + x_j^i\right) = \sum_{\substack{i,j\in N\\ i<j}} (a_{ij}+a_{ji}) = v_A(N).$$

Analogously, for each $S \subset N$,

$$\sum_{i\in S} x_i = \sum_{i\in S}\sum_{j\in N\setminus\{i\}} x_i^j \geq \sum_{i\in S}\sum_{j\in S\setminus\{i\}} x_i^j = \sum_{\substack{i,j\in S\\ i<j}} \left(x_i^j + x_j^i\right) = \sum_{\substack{i,j\in S\\ i<j}} (a_{ij}+a_{ji}) = v_A(S).$$

Then, $x \in C(N,v_A)$.

Conversely, let $x = (x_i)_{i\in N} \in C(N,v_A)$. As (N,v_A) is convex, the core is the convex hull of the vector of marginal contributions. Thus, there exists $(y_\pi)_{\pi\in\Pi_N}$ with $y_\pi \geq 0$ for each $\pi \in \Pi_N$ and $\sum\limits_{\pi\in\Pi_N} y_\pi = 1$ such that, for each $i \in N$,

$$x_i = \sum_{\pi\in\Pi_N} y_\pi \left[v_A\left(Pre(i,\pi)\cup\{i\}\right) - v_A\left(Pre(i,\pi)\right)\right].$$

[14]A Terrestrial Flight Telephone System refers to an agreement made by a group of countries in order to provide a network of ground stations so that phone calls can be made within their airplanes while flying above their territory.

Because of the definition of v_A, we have that

$$x_i = \sum_{\pi \in \Pi_N} y_\pi \left[\sum_{j \in Pre(i,\pi)} (a_{ij} + a_{ji}) \right] = \sum_{j \in N \setminus \{i\}} (a_{ij} + a_{ji}) \sum_{\pi \in \Pi_N, j \in Pre(i,\pi)} y_\pi .$$

For each pair $i, j \in N$, with $i \neq j$, we define

$$x_i^j = (a_{ij} + a_{ji}) \sum_{\pi \in \Pi_N, j \in Pre(i,\pi)} y_\pi .$$

Thus, $x_i^j \geq 0$, for each $j \in N \setminus \{i\}$, and for each $i \in N$, i.e., (i) holds.

Furthermore, $\sum_{j \in N \setminus \{i\}} x_i^j = x_i$, i.e., (ii) holds.

Let $i, j \in N$ with $i \neq j$. Then,

$$\begin{aligned} x_i^j + x_j^i &= \left((a_{ij} + a_{ji}) \sum_{\pi \in \Pi_N, j \in Pre(i,\pi)} y_\pi \right) + \left((a_{ij} + a_{ji}) \sum_{\pi \in \Pi_N, i \in Pre(j,\pi)} y_\pi \right) \\ &= (a_{ij} + a_{ji}) \sum_{\pi \in \Pi_N} y_\pi = a_{ij} + a_{ji}, \end{aligned}$$

i.e., (iii) holds. ■

The above proposition states that, in order to satisfy the core constraints, we should divide the revenue generated by the audience of a game between the two teams playing the game. There is complete freedom within those bounds. For instance, assigning all the revenue to one of the teams would be admissible. The Shapley rule states that the revenue generated by the audience of a game be divided equally between both teams. Thus, the allocations that the Shapley rule yields satisfy the core constraints, as mentioned above.

This is a strong argument to endorse the Shapley rule. Teams are corporations and, as such, any subgroup of teams could potentially secede and form another (smaller) competition. Thus, if the rule selects allocations within the core, it provides stable outcomes, in the sense of dismissing incentives for team secessions. As shown above, in this case, the core is non-empty and very large. Thus, it seems reasonable to select one allocation within the core.

3.2 The conflicting claims approach

O'Neill (1982) is credited for introducing one of the simplest (and yet useful) models to study distributive justice. The so-called problem of *adjudicating conflicting claims* refers to a situation in which an insufficient amount of a perfectly divisible good (endowment) has to be allocated among a group of agents who hold claims against it. This basic framework is flexible enough to accommodate a variety of related situations that trace back to ancient sources such as Aristotle's essays and the Talmud.[15] It turns out that, as we show in this section, our problems could also be seen as a specific instance of the problem of adjudicating conflicting claims.

Formally, a *problem of adjudicating conflicting claims* (or, simply, a claims problem) is a triple consisting of a population $N \in \mathcal{N}$, a claims profile $c \in \mathbb{R}^n_+$, and an *endowment* $E \in \mathbb{R}_+$ such that $\sum_{i\in N} c_i \geq E$. Let $C \equiv \sum_{i\in N} c_i$. To avoid unnecessary complication, we assume $C > 0$. Let $\mathcal{D}^N$ be the domain of bankruptcy problems with population N and $\mathcal{D} \equiv \bigcup_{N\in\mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, c, E) \in \mathcal{D}^N$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i\in N} x_i = E$. We refer to (i) as *boundedness* and (ii) as *balance*. A *rule* on $\mathcal{D}$, $R \colon \mathcal{D} \to \bigcup_{N\in\mathcal{N}} \mathbb{R}^n$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $R\,(N, c, E)$ for the problem.

The so-called *constrained equal-awards* rule, CEA, selects, for each $(N, c, E) \in \mathcal{D}$, the vector $(\min\{c_i, \lambda\})_{i\in N}$, where $\lambda > 0$ is chosen so that $\sum_{i\in N} \min\{c_i, \lambda\} = E$.

The so-called *constrained equal-losses* rule, CEL, selects, for each $(N, c, E) \in \mathcal{D}$, the vector $(\max\{0, c_i - \lambda\})_{i\in N}$, where $\lambda > 0$ is chosen so that $\sum_{i\in N} \max\{0, c_i - \lambda\} = E$.

The so-called *Talmud* rule is a hybrid between the above two. More precisely, for each $(N, c, E) \in \mathcal{D}$, it selects

$$T\,(N, c, E) = \begin{cases} CEA(N, \frac{1}{2}c, E) & \text{if } E \leq \frac{1}{2}C \\ \frac{1}{2}c + CEL(N, \frac{1}{2}c, E - \frac{1}{2}C) & \text{if } E \geq \frac{1}{2}C \end{cases}$$

Finally, the so-called *proportional* rule, P, yields awards proportionally to claims, i.e., for each $(N, c, E) \in \mathcal{D}$, $P(N, c, E) = \frac{E}{C} \cdot c$.

In a (broadcasting sports) problem (N, A), as formalized above, the issue is to allocate the aggregate audience in the tournament $(||A||)$ among the participating teams (N). If one considers each team claims the overall audience of the games it was involved $(\alpha_i\,(A))$, then we obviously have a problem of adjudicating conflicting claims. More precisely, we associate with each (broadcasting sports) problem (N, A) a claims problem $\left(N, c^A, E^A\right)$ where $c^A_i = \alpha_i\,(A)$, for each $i \in N$, and $E^A = ||A||$.

[15]The reader is referred to Thomson (2003, 2015, 2017) for excellent surveys of the sizable literature dealing with this model.

In Example 1 we have that $E = 4920$ and

i	1	2	3
c_i^A	4460	2860	2520
$P_i\left(N, c^A, E^A\right)$	2230	1430	1260
$CEA_i\left(N, c^A, E^A\right)$	1640	1640	1640
$CEL_i\left(N, c^A, E^A\right)$	2820	1220	880
$T_i\left(N, c^A, E^A\right)$	2230	1430	1260

The next result summarizes our main findings. The Talmud rule coincides with the proportional rule and the Shapley rule. The CEA rule and the CEL rule do not guarantee allocations within the core. It also states the stability properties of the above rules. It turns out that only the proportional rule (or the Talmud rule, as they both coincide in this setting) guarantees allocations within the core. This is due to the fact that, as mentioned above, the proportional rule yields the same outcomes as the Shapley rule.

Proposition 2 *The following statements hold:*

(a) $P\left(N, c^A, E^A\right) = T\left(N, c^A, E^A\right) = S(N, A) \in C(N, v_A)$, *for each* $(N, A) \in \mathcal{P}$.

(b) $CEA\left(N, c^A, E^A\right) \notin C(N, v_A)$ *for some* $(N, A) \in \mathcal{P}$.

(c) $CEL\left(N, c^A, E^A\right) \notin C(N, v_A)$ *for some* $(N, A) \in \mathcal{P}$.

Proof. (a) As $E^A = \frac{C^A}{2}$ we have that

$$P\left(N, c^A, E^A\right) = T\left(N, c^A, E^A\right) = \frac{c}{2}.$$

Thus, $P\left(N, c^A, E^A\right) = T\left(N, c^A, E^A\right) = S(N, A)$, for each $(N, A) \in \mathcal{P}$. We have seen above that $S(N, A) \in C(N, v_A)$, for each $(N, A) \in \mathcal{P}$.

(b) Let $(N, A) \in \mathcal{P}$ be such that $N = \{1, 2, 3, 4\}$ and

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then, $E^A = 8$, $c^A = (6, 6, 2, 2)$. Thus, $CEA\left(N, c^A, E^A\right) = (2, 2, 2, 2)$. As

$$CEA_1\left(N, c^A, E^A\right) + CEA_2\left(N, c^A, E^A\right) = 4 < 6 = a_{12} + a_{21},$$

it follows from Proposition 1 (b) that $CEA\left(N, c^A, E^A\right) \notin C\left(N, v_A\right)$.

(c) Let $(N, A) \in \mathcal{P}$ be such that $N = \{1, 2, 3, 4\}$ and

$$A = \begin{pmatrix} 0 & 9 & 0 & 0 \\ 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then, $E^A = 20$, $c^A = (18, 18, 2, 2)$ and $CEL\left(N, c^A, E^A\right) = (10, 10, 0, 0)$. As

$$CEL_3\left(N, c^A, E^A\right) + CEL_4\left(N, c^A, E^A\right) = 0 < 2 = a_{34} + a_{43},$$

it follows from Proposition 1 (b) that $CEL\left(N, c^A, E^A\right) \notin C\left(N, v_A\right)$ ■

4 The optimistic scenario for the fan effect and the concede-and-divide rule

In this section we consider the opposite scenario to that analyzed in the previous section. More precisely, we assume that nobody belongs to group 4, i.e., nobody is a joint fan of both teams in a game. In other words, we assume that when somebody decides to watch a game, it is because he/she is a fan of one of the teams or because he/she is a fan of the specific sport being considered. In this scenario, we believe each team should receive the revenues generated by its fans, whereas the revenue coming from the generic sport fans should be divided equally among all teams. We may argue that this optimistic scenario is extreme and also that the pessimistic scenario analyzed in the previous section was extreme in the opposite way. It will be nevertheless interesting to understand the two polar cases as they will represent meaningful lower and upper bounds for the amounts teams should get.

Let b_0 denote the number of generic sport fans. For each $i \in N$, let b_i denote the number of fans of team i. Thus, for each pair $i, j \in N$, with $i \neq j$.

$$a_{ij} = b_0 + b_i + b_j + \varepsilon_{ij}.$$

Viewers from group 4 are therefore collected in $\{\varepsilon_{ij}\}_{i,j \in N}$. As in this scenario we are assuming that nobody belongs to group 4, our aim is to make $\{\varepsilon_{ij}\}_{i,j \in N}$ as small as we can (given the audience data).

Thus, we take $b = \{b_i\}_{i=0}^{n} \in \mathbb{R}^{n+1}$ such that

$$\min_{b\in\mathbb{R}^{n+1}} \sum_{i,j\in N, i\neq j} \varepsilon_{ij}^2 = \min_{b\in\mathbb{R}^{n+1}} \sum_{i,j\in N, i\neq j} \left(a_{ij} - b_0 - b_i - b_j\right)^2. \tag{1}$$

Let $\hat{b}_0$ and $\left\{\hat{b}_i\right\}_{i\in N}$ denote the solutions to (1). Besides, for each pair $i, j \in N$, $i \neq j$ we denote

$$\hat{\varepsilon}_{ij} = a_{ij} - \widehat{b_0} - \widehat{b_i} - \widehat{b_j}$$

We now divide the audience according to the following principles:

$(P1)$ $\hat{b}_0$ is divided equally among all teams.[16]

$(P2)$ $\hat{b}_i$ is assigned to team i.

$(P3)$ $\hat{\varepsilon}_{ij}$ is divided equally between teams i and j.

Applying those principles we can define a rule $R^b(N, A)$ where, for each problem $(N, A) \in \mathcal{P}$ and each $i \in N$, the allocation for team i is[17]

$$R_i^b(N, A) = (n-1)\widehat{b_0} + 2(n-1)\widehat{b_i} + \sum_{j\in N\setminus\{i\}} \frac{\widehat{\varepsilon_{ij}} + \widehat{\varepsilon_{ji}}}{2}. \tag{2}$$

Unfortunately, the minimization problem (1) cannot be solved.[18] We then remove one of the teams $k \in N$, and consider the following minimization problem. We take $b = \left\{b_0, \{b_i\}_{i\in N\setminus\{k\}}\right\} \in \mathbb{R}^n$ such that

$$\min_{b\in\mathbb{R}^{n}} \sum_{i,j\in N, i\neq j} \varepsilon_{ij}^2 \tag{3}$$

where

$$\varepsilon_{ij} = \begin{cases} a_{ij} - b_0 - b_i - b_j & \text{if } k \notin \{i, j\} \\ a_{ij} - b_0 - b_i & \text{if } k = j \\ a_{ij} - b_0 - b_j & \text{if } k = i \end{cases}$$

[16] If, instead, we assume that it is divided equally between teams i and j, nothing will change.

[17] Note that each team plays $2(n-1)$ games.

[18] This is due to the fact that the minimization problem (1) coincides with the minimization problem induced by the OLS estimator associated with the following regression model, which involves colinearity:

$$Y = b_0 + \sum_{i\in N} b_i X_i + \varepsilon,$$

where Y is the audience of a game, X_i is the team dummy variable (i.e., $X_i = 1$ if team i plays the game and 0 otherwise) and ε is the error term. It is straightforward to see that, for each $k = 1, ..., n$, $X_k = 2_A - \sum_{i\in N\setminus\{k\}} X_i$, where 2_A is the vector with all coordinates equal to 2.

Then, we can apply principles $(P1)$, $(P2)$ and $(P3)$ to the solution of this problem in order to obtain a rule as in (2). The potential issue that could arise is that the derived allocation would depend on k (the removed team). The next theorem shows that this is not the case.

Theorem 2 *For each (N, A) and each pair $i, k \in N$, let $R_i^{b,k}(N, A)$ be the allocation obtained by applying formula (2) to the minimization problem (3). Then,*

$$R_i^{b,k}(N, A) = \frac{(n-1)\,\alpha_i - ||A||}{n-2} = CD_i(N, A).$$

Proof. We note first that the solution to the minimization problem (3) coincides with the OLS estimator of the linear regression model where the set of dependent variables is $\{X_i\}_{i \in N \setminus \{k\}}$, and thus, no colinearity occurs.

Given the linear regression model $V = b_0 + \sum_{i \in S} b_i U_i + \varepsilon$, it is well known that the OLS estimator is computed as

$$\begin{aligned} \left(\widehat{b_i}\right)_{i \in S} &= Cov(U, U)^{-1}\, Cov(U, V) \text{ and} \\ \widehat{b_0} &= \overline{V} - \sum_{i \in S} \widehat{b_i}\overline{U_i} \end{aligned} \tag{4}$$

where

$$\begin{aligned} Cov(U, U) &= \left(Cov(U_i, U_j)\right)_{i,j \in S} \text{ and} \\ Cov(U, V) &= \left(Cov(U_i, V)\right)_{i \in S} \end{aligned}$$

Besides, given two variables U, V taking the values $\{(u_k, v_k)\}_{k=1}^{m}$ we have that

$$Cov(U, V) = \frac{\sum_{k=1}^{m} u_k v_k}{m} - \left(\frac{\sum_{k=1}^{m} u_k}{m}\right)\left(\frac{\sum_{k=1}^{m} v_k}{m}\right).$$

We now apply the previous expressions to our case.

1. Let $i, j \in N$ with $i \neq j$.

$$\begin{aligned} Cov(X_i, X_j) &= \frac{2}{n(n-1)} - \left(\frac{2(n-1)}{n(n-1)}\right)\left(\frac{2(n-1)}{n(n-1)}\right) \\ &= \frac{2}{n(n-1)} - \frac{4}{n^2} = \frac{2(2-n)}{n^2(n-1)}. \end{aligned}$$

2. Let $i \in N$.

$$\begin{aligned} Cov\left(X_i, X_i\right) &= \frac{2\left(n-1\right)}{n\left(n-1\right)} - \left(\frac{2\left(n-1\right)}{n\left(n-1\right)}\right)\left(\frac{2\left(n-1\right)}{n\left(n-1\right)}\right) \\ &= \frac{2}{n} - \frac{4}{n^2} = \frac{2\left(n-2\right)}{n^2}. \end{aligned}$$

3. Let $i \in N$.

$$\begin{aligned} Cov\left(X_i, Y\right) &= \frac{\alpha_i}{n\left(n-1\right)} - \left(\frac{2\left(n-1\right)}{n\left(n-1\right)}\right)\left(\frac{\left|\left|A\right|\right|}{n\left(n-1\right)}\right) \\ &= \frac{\alpha_i}{n\left(n-1\right)} - \frac{2\left|\left|A\right|\right|}{n^2\left(n-1\right)} \\ &= \frac{n\alpha_i - 2\left|\left|A\right|\right|}{n^2\left(n-1\right)} = \left(\alpha_i - \frac{2\left|\left|A\right|\right|}{n}\right)\frac{1}{n\left(n-1\right)}. \end{aligned}$$

Then $Cov\left(U, U\right) = Cov\left(X_i, X_j\right)_{i,j \in N \setminus \{k\}}$ is a matrix of $\left(n-1\right) \times \left(n-1\right)$ dimension. It is not difficult to show that

$$Cov\left(U, U\right)^{-1} = \frac{n\left(n-1\right)}{2\left(n-2\right)}\begin{pmatrix} 2 & 1 & ... & 1 \\ 1 & 2 & ... & 1 \\ 1 & ... & ... & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \tag{5}$$

Besides,

$$Cov\left(U, V\right) = \frac{1}{n^2\left(n-1\right)}\begin{pmatrix} n\alpha_1 - 2\left|\left|A\right|\right| \\ ... \\ n\alpha_n - 2\left|\left|A\right|\right| \end{pmatrix} \tag{6}$$

Because of (4), we have that, for each $j \in N \setminus \{k\}$,

$$\begin{aligned} \widehat{b_j} &= \frac{n\left(n-1\right)}{2\left(n-2\right)}\frac{1}{n^2\left(n-1\right)}\left[2\left(n\alpha_j - 2\left|\left|A\right|\right|\right) + \sum_{i \in N \setminus \{j,k\}}\left(n\alpha_i - 2\left|\left|A\right|\right|\right)\right] \\ &= \frac{1}{2\left(n-2\right)n}\left[2n\alpha_j - 4\left|\left|A\right|\right| + n\sum_{i \in N \setminus \{j,k\}}\alpha_i - 2\left(n-2\right)\left|\left|A\right|\right|\right] \\ &= \frac{1}{2\left(n-2\right)n}\left[2n\alpha_j + n\sum_{i \in N \setminus \{j,k\}}\alpha_i - 2n\left|\left|A\right|\right|\right]. \end{aligned}$$

As $\sum_{i\in N} \alpha_i = 2\,||A||$, we have that

$$\begin{aligned}
\widehat{b_j} &= \frac{1}{2(n-2)n}\left[2n\alpha_j + n\left(2\,||A|| - (\alpha_j+\alpha_k)\right) - 2n\,||A||\right] \\
&= \frac{1}{2(n-2)n}\left[2n\alpha_j + 2n\,||A|| - n(\alpha_j+\alpha_k) - 2n\,||A||\right] \\
&= \frac{1}{2(n-2)n}\left[n(\alpha_j-\alpha_k)\right] = \frac{\alpha_j-\alpha_k}{2(n-2)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\widehat{b_0} &= \overline{Y} - \sum_{j\in N\setminus\{k\}} \widehat{b_j}\overline{X_j} = \frac{||A||}{n(n-1)} - \sum_{j\in N\setminus\{k\}} \frac{\alpha_j-\alpha_k}{2(n-2)}\frac{2(n-1)}{n(n-1)} \\
&= \frac{||A||}{n(n-1)} - \sum_{j\in N\setminus\{k\}} \frac{\alpha_j-\alpha_k}{n(n-2)} \\
&= \frac{||A||}{n(n-1)} - \frac{1}{n(n-2)}\left[\sum_{j\in N\setminus\{k\}} \alpha_j - (n-1)\alpha_k\right] \\
&= \frac{||A||}{n(n-1)} - \frac{1}{n(n-2)}\left[2\,||A|| - \alpha_k - (n-1)\alpha_k\right] \\
&= \frac{||A||}{n(n-1)} - \frac{2\,||A||}{n(n-2)} + \frac{\alpha_k}{n-2} = -\frac{||A||}{(n-1)(n-2)} + \frac{\alpha_k}{n-2}.
\end{aligned}$$

Once we have estimated the parameters we have that

$$\begin{aligned}
a_{ij} = \widehat{b_0} + \widehat{b_i} + \widehat{b_j} + \widehat{\varepsilon_{ij}} &\quad \text{if } i,j\in N\setminus\{k\} \\
a_{ik} = \widehat{b_0} + \widehat{b_i} + \widehat{\varepsilon_{ik}} &\quad \text{if } i\in N\setminus\{k\} \\
a_{ki} = \widehat{b_0} + \widehat{b_i} + \widehat{\varepsilon_{ki}} &\quad \text{if } i\in N\setminus\{k\}.
\end{aligned}$$

Given $i,j\in N\setminus\{k\}$,

$$\begin{aligned}
\widehat{\varepsilon_{ij}} &= a_{ij} - \widehat{b_0} - \widehat{b_i} - \widehat{b_j} = \\
&= a_{ij} + \frac{||A||}{(n-1)(n-2)} - \frac{\alpha_k}{n-2} - \frac{\alpha_i-\alpha_k}{2(n-2)} - \frac{\alpha_j-\alpha_k}{2(n-2)} \\
&= a_{ij} + \frac{||A||}{(n-1)(n-2)} - \frac{\alpha_i+\alpha_j}{2(n-2)}.
\end{aligned}$$

Given $i\in N\setminus\{k\}$,

$$\begin{aligned}
\widehat{\varepsilon_{ik}} &= a_{ik} - \widehat{b_0} - \widehat{b_i} = \\
&= a_{ik} + \frac{||A||}{(n-1)(n-2)} - \frac{\alpha_k}{n-2} - \frac{\alpha_i-\alpha_k}{2(n-2)} \\
&= a_{ik} + \frac{||A||}{(n-1)(n-2)} - \frac{\alpha_i+\alpha_k}{2(n-2)}.
\end{aligned}$$

Analogously, we have that

$$\widehat{\varepsilon_{ki}} = a_{ki} + \frac{||A||}{(n-1)(n-2)} - \frac{\alpha_i + \alpha_k}{2(n-2)}.$$

Notice that, for each pair $i, j \in N$,

$$\widehat{\varepsilon_{ij}} = a_{ij} + \frac{||A||}{(n-1)(n-2)} - \frac{\alpha_i + \alpha_j}{2(n-2)}. \tag{7}$$

We now compute the rule $R_i^{b,k}(N, A)$ by applying principles $(P1)$, $(P2)$ and $(P3)$ in this regression. We consider two cases:

- Team $i \in N \setminus \{k\}$. The audience assigned to team i is made of three components:

  By $(P1)$, team i receives

$$(n-1)\widehat{b_0} = -\frac{||A||}{n-2} + \frac{(n-1)\alpha_k}{n-2}.$$

  By $(P2)$, team i receives

$$2(n-1)\widehat{b_i} = \frac{(n-1)(\alpha_i - \alpha_k)}{n-2}.$$

  By $(P3)$, team i receives

$$\begin{aligned}
\sum_{j \in N \setminus \{i\}} \frac{\widehat{\varepsilon_{ij}} + \widehat{\varepsilon_{ji}}}{2} &= \frac{1}{2} \sum_{j \in N \setminus \{i\}} (a_{ij} + a_{ji}) + \frac{||A||}{(n-2)} - \frac{(n-1)\alpha_i + \sum_{j \in N \setminus \{i\}} \alpha_j}{2(n-2)} \\
&= \frac{\alpha_i}{2} + \frac{||A||}{n-2} - \frac{(n-1)\alpha_i + 2||A|| - \alpha_i}{2(n-2)} \\
&= \frac{\alpha_i}{2} + \frac{||A||}{(n-2)} - \frac{\alpha_i}{2} - \frac{||A||}{n-2} = 0.
\end{aligned}$$

  Thus, team i receives

$$\begin{aligned}
R_i^{b,k}(N, A) &= -\frac{||A||}{n-2} + \frac{(n-1)\alpha_k}{n-2} + \frac{(n-1)(\alpha_i - \alpha_k)}{n-2} \\
&= \frac{(n-1)\alpha_i - ||A||}{n-2}.
\end{aligned}$$

- Team k. The audience assigned to team k is also made of three components:

  By $(P1)$, team k receives

$$(n-1)\widehat{b_0} = -\frac{||A||}{n-2} + \frac{(n-1)\alpha_k}{n-2}.$$

By $(P2)$, team k receives nothing.

Analogously to the previous case, by $(P3)$, team k receives nothing.

Thus, team k receives

$$R_k^{P,k}(N,A) = \frac{(n-1)\,\alpha_k - ||A||}{n-2}.$$

■

Theorem 2 shows that the rule derived from (2), with the minimization problem (3), is precisely the concede-and-divide rule introduced in Section 2.

5 The axiomatic approach

The previous two sections provided arguments to endorse, respectively, the two focal rules of this work. First, the Shapley rule was shown to coincide with the Shapley value (as well as the Nucleolus and the τ-value) of a natural convex TU-game, thus guaranteeing stable outcomes (as formalized by the core of such a game). Second, the concede-and-divide rule arose as the outcome of an optimization problem aiming to minimize the number of *indifferent fans*. In this section, we provide normative foundations for both rules. We first give a list of axioms and study which ones are satisfied by each rule. Later on we characterize both rules using some of them.

The first axiom we consider says that if two teams have the same audiences, then they should receive the same amount.

Equal treatment of equals: For each $(N,A) \in \mathcal{P}$, and each pair $i,j \in N$ such that $a_{ik} = a_{jk}$, and $a_{ki} = a_{kj}$, for each $k \in N \setminus \{i,j\}$,

$$R_i(N,A) = R_j(N,A).$$

The second axiom says that revenues should be additive on A. Formally,

Additivity: For each pair (N,A) and $(N,A') \in \mathcal{P}$

$$R(N,A+A') = R(N,A) + R(N,A').$$

The third axiom says that if nobody watches a single game of a given team (i.e., the team has a null audience), then such a team gets no revenue.

Null team: For each $(N, A) \in \mathcal{P}$, and each $i \in N$, such that $a_{ij} = 0 = a_{ji}$, for each $j \in N$,

$$R_i(N, A) = 0.$$

Alternatively, the next axiom says that if a team nullifies the audience of all the games it plays (for instance, due to some kind of boycott), then the allocation of such a team should decrease exactly by the total audience of such a team.[19] Formally,

Nullifying team: For each $(N, A), (N, A') \in \mathcal{P}$ such that there exists $k \in N$ (the nullifying team) satisfying $a'_{ij} = a_{ij}$ when $k \notin \{i, j\}$ and $a'_{ij} = 0$ when $k \in \{i, j\}$ we have that

$$R_k(N, A') = R_k(N, A) - \alpha_k(A).$$

The next axiom says that the allocation should be in the core of the game v_A, described in Section 3.

Core selection: For each $(N, A) \in \mathcal{P}$,

$$R(N, A) \in C\left(N, v_A\right).$$

The next axiom says that no team should receive negative awards.

Non negativity. For each $(N, A) \in \mathcal{P}$ and $i \in N$,

$$R_i\left(N, A\right) \geq 0.$$

The next axiom says that if the audience of team i is, game by game, not smaller than the audience of team j, then team i could not receive less than team j.

Monotonicity: For each $(N, A) \in \mathcal{P}$ and each pair $i, j \in N$, such that, for each $k \in N \backslash \{i, j\}$, $a_{ik} \geq a_{jk}$ and $a_{ki} \geq a_{kj}$ we have that

$$R_i(N, A) \geq R_j\left(N, A\right).$$

The next axiom says that each team should receive, at most, the total audience of the games played by the team.

[19] A similar axiom was introduced in cooperative transferable utility games by van den Brink (2007).

Maximum aspirations: For each $(N, A) \in \mathcal{P}$ and each $i \in N$,

$$R_i(N, A) \leq \alpha_i(A).$$

The next axiom refers to the incremental effect of adding a single additional viewer to a game. It states that the additional revenue should be shared equally among the involved teams in such a game. Formally,

Equal sharing of additional viewers: For each pair $(N, A), (N, \hat{A}) \in \mathcal{P}$ such that $a_{ij} = \hat{a}_{ij}$, for each pair $(i, j) \neq (i_0, j_0)$, and $a_{i_0,j_0} + 1 = \hat{a}_{i_0,j_0}$,

$$R_{i_0}(N, \hat{A}) - R_{i_0}(N, A) = R_{j_0}(N, \hat{A}) - R_{j_0}(N, A).$$

We now study which axioms are satisfied by each rule.

Proposition 3 *The Shapley rule satisfies equal treatment of equals, additivity, null team, monotonicity, core selection, non negativity, maximum aspirations, and equal sharing of additional viewers, but violates nullifying team.*

Proof. It is trivial to show that S satisfies equal treatment of equals, null team, monotonicity, non negativity, and maximum aspirations. We have already seen that S satisfies core selection. Additivity is a consequence of the fact that, for each pair $(N, A), (N, A') \in \mathcal{P}$, and each $i \in N$, $\alpha_i(N, A + A') = \alpha_i(N, A) + \alpha_i(N, A')$. Similarly, equal sharing of additional viewers is a consequence of the fact that $\alpha_i(N, \hat{A}) = \alpha_i(N, A) + 1$ when $i \in \{i_0, j_0\}$.

As for nullifying team, let (N, A) be such that $N = \{1, 2, 3\}$ and $a_{ij} = 10$ for each pair $i, j \in N$, $i \neq j$. Let (N, A') be obtained from A by nullifying team 3. Namely, $a'_{12} = a'_{21} = 10$ and $a'_{ij} = 0$ otherwise. Then $\alpha(A) = (40, 40, 40)$ and $\alpha(A') = (20, 20, 0)$. Hence, $S(N, A) = (20, 20, 20)$ and $S(N, A') = (10, 10, 0)$, which shows that S does not satisfy nullifying team, as $S_3(N, A') = 0 \neq -20 = S_3(N, A) - \alpha_3(A)$. ■

Proposition 4 *The concede-and-divide rule satisfies equal treatment of equals, additivity, nullifying team, monotonicity, maximum aspirations, and equal sharing of additional viewers, but violates null team, core selection and non negativity.*

Proof. It is trivial to show that CD satisfies equal treatment of equals, monotonicity and maximum aspirations. Additivity is a consequence of the fact that, for each pair $(N,A),(N,A')\in\mathcal{P}$, and each $i\in N$, $||A+A'||=||A||+||A'||$ and $\alpha_i(N,A+A')=\alpha_i(N,A)+\alpha_i(N,A')$.

Let $(N,A),(N,A')\in\mathcal{P}$ and $k\in N$ be as in the definition of nullifying team. Then,

$$\begin{aligned} CD_k(N,A') &= \alpha_k(N,A')-\frac{\sum\limits_{j,k\in N\setminus\{i\}}\left(a'_{jk}+a'_{kj}\right)}{n-2} \\ &= -\frac{\sum\limits_{j,k\in N\setminus\{i\}}\left(a_{jk}+a_{kj}\right)}{n-2} \\ &= \alpha_k(N,A)-\frac{\sum\limits_{j,k\in N\setminus\{i\}}\left(a_{jk}+a_{kj}\right)}{n-2}-\alpha_k(N,A) \\ &= CD_k(N,A)-\alpha_k(N,A). \end{aligned}$$

Then, CD satisfies nullifying team.

Let $(N,A),\left(N,\hat{A}\right)\in\mathcal{P}$ and (i_0,j_0) as in the definition of equal sharing of additional viewers. As $\left|\left|\hat{A}\right|\right|=||A||+1$ and $\alpha_i\left(N,\hat{A}\right)=\alpha_i(N,A)+1$ when $i\in\{i_0,j_0\}$, we have that, for each $i\in\{i_0,j_0\}$,

$$\begin{aligned} R_i\left(N,\hat{A}\right)-R_i(N,A) &= \frac{(n-1)\,\alpha_i\left(N,\hat{A}\right)-\left|\left|\hat{A}\right|\right|}{n-2}-\frac{(n-1)\,\alpha_i(N,A)-||A||}{n-2} \\ &= \frac{(n-1)\left[\alpha_i(N,A)+1\right]-\left[||A||+1\right]}{n-2}-\frac{(n-1)\,\alpha_i(N,A)-||A||}{n-2} \\ &= 1. \end{aligned}$$

Then, CD satisfies equal sharing of additional viewers.

As for the remaining axioms, let $(N,A)\in\mathcal{P}$ be such that $N=\{1,2,3\}$, $a_{12}=a_{21}=10$ and $a_{ij}=0$ otherwise. Then, $||A||=20$ and $\alpha=(20,20,0)$. Hence $CD(N,A)=(20,20,-20)$. From here, it follows that CD does not satisfy null team, because $a_{3i}=a_{i3}=0$, for each $i\in N$, but $CD_3(N,a)=-20\neq 0$. Similarly, CD does not satisfy non negativity because $CD_3(N,a)=-20<0$, and core selection because $CD_3(N,A)=-20<0=v_A(3)$. ■

We have seen that CD could provide negative awards. This fact is not counterintuitive at all in the optimistic scenario for the fan effect. Consider, for instance, a league with three teams where team 1 has 600 followers, team 2 has also 600, and team 3 has not followers. Besides, no follower of team 1 and 2 wants to watch the games of team 3. This situation induces a problem where $a_{12}=a_{21}=1200$ and $a_{ij}=0$ otherwise. Under this assumption, team 1 should receive

2400 because it plays four games in the league and with a marginal contribution of 600 fans to each game. The same happens with team 2. Nevertheless, the contribution of team 3 to the league is negative because it has no fans and, moreover, the other teams' fans do now want to watch the games of team 3.

The next result provides the characterizations of the two rules.

Theorem 3 *The following statements hold:*

(a) *A rule satisfies equal treatment of equals, additivity and null team if and only if it is the Shapley rule.*

(b) *A rule satisfies equal sharing of additional viewers, additivity and null team if and only if it is the Shapley rule.*

(c) *A rule satisfies equal treatment of equals and nullifying team if and only if it is the concede-and-divide rule.*

Proof. (a) By Proposition 3, the Shapley rule satisfies the three axioms.

Conversely, let $(N, A) \in \mathcal{P}$. For each pair $i, j \in N$, with $i \neq j$, let A^{ij} denote the matrix with the following entries:

$$a_{kl}^{ij} = \begin{cases} a_{ij} & \text{if } (k,l) = (i,j) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $a_{ji}^{ij} = 0$.

Let $k \in N$. By additivity,

$$R_k(N, A) = \sum_{i,j \in N : i \neq j} R_k\left(N, A^{ij}\right).$$

By null team, for each pair $i, j \in N$ with $i \neq j$, and for each $l \in N \backslash \{i, j\}$, we have $R_l(N, A^{ij}) = 0$. Thus,

$$R_k(N, A) = \sum_{l \in N \backslash \{k\}} \left[R_k\left(N, A^{lk}\right) + R_k\left(N, A^{kl}\right)\right].$$

By equal treatment of equals, $R_k\left(N, A^{lk}\right) = R_l\left(N, A^{lk}\right)$. As $\left|\left|A^{lk}\right|\right| = a_{lk}$, we have that $R_k\left(N, A^{lk}\right) = \frac{a_{lk}}{2}$. Similarly, $R_k\left(N, A^{kl}\right) = \frac{a_{kl}}{2}$. Thus,

$$R_k(N, A) = \sum_{l \in N \backslash \{k\}} \left[\frac{a_{lk}}{2} + \frac{a_{kl}}{2}\right] = \frac{\alpha_k}{2} = S_k(N, A).$$

(b) By Proposition 3, the Shapley rule satisfies the three axioms.

Conversely, let $(N, A) \in \mathcal{P}$ and $k \in N$. As in the proof of (a), it follows, by additivity and null team, that

$$R_k(N, A) = \sum_{l \in N \setminus \{k\}} \left[R_k\left(N, A^{lk}\right) + R_k\left(N, A^{kl}\right) \right].$$

Let $0_{N,N}$ be the matrix with the same dimension of A and all entries equal to 0. By null team, $R_j(N, 0_{N,N}) = 0$, for each $j \in N$. By equal sharing of additional viewers,

$$\begin{aligned} R_k\left(N, A^{lk}\right) &= R_k\left(N, A^{lk}\right) - R_k(N, 0_{N,N}) \\ &= R_l\left(N, A^{lk}\right) - R_l(N, 0_{N,N}) \\ &= R_l\left(N, A^{lk}\right). \end{aligned}$$

From here, an analogous argument to that in the proof of (a) allows to deduce too that $R(N, A) = S(N, A)$.

(c) By Proposition 4, the concede-and-divide rule satisfies both axioms.

Conversely, let R be a rule satisfying the two axioms in the statement. Let $(N, A) \in \mathcal{P}$. Let $t(A)$ be the number of null teams in (N, A). We procceed recursively on $t(A)$. Notice that $t(A) \in \{0, 1, ..., n-2, n\}$.

Suppose first that $t(A) = n$. Then, $A = 0_{N,N}$ (the matrix with all entries equal to 0). By equal treatment of equals, for each $i \in N$,

$$R_i(N, 0_{N,N}) = 0 = CD_i(N, 0_{N,N}).$$

Suppose now that $t(A) = n - 2$. Then $A = A^{ij} + A^{ji}$ for some $i, j \in N$ and hence, $\alpha_i(A) = \alpha_j(A) = a_{ij} + a_{ji}$ and $\alpha_k(A) = 0$ otherwise. Then,

$$CD_k(N, A) = \begin{cases} a_{ij} + a_{ji} & \text{if } k \in \{i, j\} \\ \frac{-(a_{ij}+a_{ji})}{n-2} & \text{otherwise.} \end{cases}$$

As (N, A), $(N, 0_{N,N})$, and $k = i$ are under the hypothesis of nullifying team,

$$0 = R_i(N, 0_{N,N}) = R_i(N, A) - (a_{ij} + a_{ji}).$$

Thus, $R_i(N, A) = a_{ij} + a_{ji}$. Analogously, we can prove that $R_j(N, A) = a_{ij} + a_{ji}$.

By equal treatment of equals, we have that $R_k(N, A^{ij}) = R_l(N, A^{ij})$, for each pair $k, l \in N \setminus \{i, j\}$. Let x denote such an amount. Thus,

$$a_{ij} + a_{ji} = ||A|| = \sum_{k \in N} R_k(N, A) = 2(a_{ij} + a_{ji}) + (n-2)x,$$

from where it follows that $x = \frac{-(a_{ij}+a_{ji})}{n-2}$.

Then, $R(N,A) = CD(N,A)$, in this case too.

Assume now that R coincides with CD in problems with r null players. We prove that both rules also coincide when we have $r-1$ null players.

Let (N,A) be a problem with $r-1$ null players. Let k be a no null player in (N,A). Let $\left(N,A^{-k}\right)$ be the problem obtained from A by nullifying team k. Namely $a_{ij}^{-k} = a_{ij}$ when $k \notin \{i,j\}$ and $a_{ij}^{-k} = 0$ when $k \in \{i,j\}$. As A and A^{-k} are under the hypothesis on the axiom of nullifying team, we deduce that

$$R_k(N,A) = R_k\left(N,A^{-k}\right) + \alpha_k(A) \text{ and } CD_k(N,A) = CD_k\left(N,A^{-k}\right) + \alpha_k(A).$$

As k is a null player in $\left(N,A^{-k}\right)$, and (N,A) has $r-1$ null players, $\left(N,A^{-k}\right)$ has r null players. As R and CD coincide in problems with r null players, we have that $R_k\left(N,A^{-k}\right) = CD_k\left(N,A^{-k}\right)$. Thus, $R_k(N,A) = CD_k(N,A)$.

Let us denote by D the set of null players in (N,A). Then,

$$\begin{aligned}\sum_{i\in D} R_i(N,A) &= ||A|| - \sum_{i\in N\setminus D} R_i(N,A)\\ &= ||A|| - \sum_{i\in N\setminus D} CD_i(N,A)\\ &= \sum_{i\in D} CD_i(N,A).\end{aligned}$$

As R and CD satisfy equal treatment of equals, all null teams in (N,A) must receive the same according to both rules. Then, for each null player i in (N,A), we have that $R_i(N,A) = CD_i(N,A)$. ■

Remark 1 *The axioms of Theorem 3 are independent.*

Let R^1 be the rule in which, for each game $(i,j) \in N \times N$, the revenue goes to the team with the lowest number of the two. Namely, for each problem $(N,A) \in \mathcal{P}$, and each $i \in N$,

$$R_i^1(N,A) = \sum_{j\in N: j>i} (a_{ij} + a_{ji}).$$

R^1 satisfies null team and additivity, but not equal treatment of equals and equal sharing of additional viewers.

The equal awards rule satisfies equal treatment of equals, equal sharing of additional viewers and additivity, but not null team.

Let R^2 be the rule that, for each pair $i, j \in N$, divides the audience a_{ij} between teams i and j proportionally to their audiences in the games played agains the other teams.[20] *Namely, for each problem $(N, A) \in \mathcal{P}$, and $i \in N$,*

$$R_i^2(N,A) = \sum_{j \in N \setminus \{i\}} \frac{\sum\limits_{k \in N \setminus \{i,j\}} (a_{ik} + a_{ki})}{\sum\limits_{k \in N \setminus \{i,j\}} (a_{ik} + a_{ki}) + \sum\limits_{k \in N \setminus \{i,j\}} (a_{jk} + a_{kj})} \left[a_{ij} + a_{ji} \right].$$

R^2 satisfies equal treatment of equals, equal sharing of additional viewers, and null team, but not additivity.

The Shapley rule satisfies equal treatment of equals but fails nullifying team.

Finally, we define the rule R^3 such that, for each problem $(N, A^{ij}) \in \mathcal{P}$, and $k \in N$,

$$R_k^3 \left(N, A^{ij} \right) = \begin{cases} a_{ij} & \text{if } k \in \{i,j\} \\ -a_{ij} & \text{if } k = \min \{ l : l \in N \setminus \{i,j\} \} \\ 0 & \text{otherwise} \end{cases}$$

We extend R^3 to all problems using additivity. Namely, $R^3(N, A) = \sum\limits_{i,j \in N : i \neq j} R^3 \left(N, A^{ij} \right)$. R^3satisfies nullifying team but fails equal treatment of equals.

Theorem 3 not only provides a characterization of our two focal rules, but also a common ground for them. More precisely, it states that both rules are characterized by the combination of equal treatment of equals, additivity, and a third axiom.[21] This different axiom (null player in one case; nullifying player in the other case) formalizes the behavior of the rule with respect to somewhat peculiar teams (those with no audience in one case; those killing audiences in the other case).

It turns out, nevertheless, that this only difference, reflected in the mentioned pair of axioms, is substantial as the axioms are incompatible. Namely, there is no rule satisfying both the null team axiom and the nullifying team axiom. Consider the problem (N, A^{12}) defined as in the proof of Theorem 3, where $N = \{1, 2, 3\}$ and $a_{12} > 0$. If R satisfies null team we have that $R_3 (N, A^{12}) = 0$ and $R_i (N, 0_{N,N}) = 0$ for each $i \in N$. Suppose that R also satisfies nullifying team. Using arguments similar to the ones used in the proof of Theorem 3 we can deduce that $R_1 (N, A^{12}) = R_2 (N, A^{12}) = a_{12}$. Thus, $R_3 (N, A^{12}) = -a_{12}$, which is a contradiction.

[20] If such other audiences are both 0, we divide equally.

[21] Actually, additivity is not necessary in the characterization of concede-and-divide.

Table 1 below summarizes the performance of both rules with respect to the axioms introduced in this section. The combination of the axioms with an asterisk in their cells characterizes the rule. The same happens for the plus symbol.

Properties	**Shapley**	**Concede-and-Divide**
Equal treatment of equals	YES*	YES*
Additivity	YES*+	YES
Null team	YES*+	NO
Nullifying team	NO	YES*
Core selection	YES	NO
Non negativity	YES	NO
Monotonicity	YES	YES
Maximum aspirations	YES	YES
Equal sharing of additional viewers	YES+	YES

Table 1: Axiomatic Analysis.

6 An empirical application

In this section, we present an empirical application of our model resorting to La Liga, the Spanish Football League.[22]

La Liga is a standard round robin tournament involving 20 teams. Thus, each team plays 38 games, facing each time one of the other 19 teams (once home, another away). The available data we have (retrieved from one of the major sport newspapers in Spain and La Liga's website) refers to the average audience of each team during the last completed season (2015-2016).[23] From there, we can derive the necessary parameters of our model; namely, the total audience

[22] http://www.laliga.es/en

[23] It is important to note that, in general, all games were broadcasted nationally in different time windows (normally, during the weekend) that did not overlap. Now, for each day (weekend) of competition, one game was broadcasted in a non-subscription channel. We do not treat those latter games distinctively in our empirical analysis. The data refers only to national broadcasting (within Spain). Although large audiences are also obtained abroad, not all games are broadcasted abroad. In order to avoid making the empirical analysis biased in favor of the teams that are more frequently broadcasted, we decided to dismiss those data from our analysis.

achieved by each team ($\alpha_i(A)$), and the aggregate audience in the league ($||A||$).[24] We also have data on the actual sharing of the revenues obtained that season. They are all collected in Table 2 below.

TEAMS	Average Audience	$\alpha_i(A)$	Revenues	%
RM	$4139, 81$	$157312, 78$	140	$14, 467$
BCN	$2739, 97$	$104118, 86$	140	$14, 467$
ATM	$1387, 43$	$52722, 34$	$69, 08$	$7, 138$
SVQ	$651, 89$	$24771, 82$	$48, 52$	$5, 014$
BET	$619, 37$	$23536, 06$	$33, 94$	$3, 507$
VAL	$582, 95$	$22152, 1$	$53, 8$	$5, 559$
CEL	$580, 92$	$22074, 96$	$33, 03$	$3, 413$
DPV	$524, 63$	$19935, 94$	$31, 68$	$3, 274$
ATH	$486, 28$	$18478, 64$	$47, 88$	$4, 948$
RVL	$473, 97$	$18010, 86$	$32, 59$	$3, 368$
RSC	$454, 72$	$17279, 36$	$38, 56$	$3, 985$
VIL	$451, 07$	$17140, 66$	$41, 72$	$4, 311$
LPA	$439, 03$	$16683, 14$	$27, 65$	$2, 857$
SPO	$417, 73$	$15873, 74$	$29, 84$	$3, 083$
MLG	$414, 32$	$15744, 16$	$38, 95$	$4, 025$
GRA	$409, 77$	$15571, 26$	$30, 99$	$3, 202$
EIB	$394, 29$	$14983, 02$	$28, 18$	$2, 912$
ESP	$384, 45$	$14609, 1$	$35, 57$	$3, 676$
LEV	$384, 07$	$14594, 66$	$33, 81$	$3, 494$
GET	$287, 52$	$10925, 76$	$31, 96$	$3, 303$

Table 2. Audiences and revenues for the Spanish Football League in 2005/2016.

Table 2 lists the 20 teams, their average audiences (in thousands), their global audiences (in thousands) and the actual revenues they made (in millions of euros), as well as in percentage terms. As we can see, two teams dominated the sharing collecting a combined 30% of the pie.

[24] Recall that $||A|| = \sum_{i \in N} \alpha_i(A)/2$.

Table 3 lists the allocations proposed by our two rules (Shapley and concede-and-divide). The numbers are normalized under the premise of our model; namely, each viewer pays a pay-per-view fee of 1 euro per game, and the overall amount is allocated. That is, $||A|| = 308259610$ euros. This is almost one third of the real revenues that the teams made combined. Thus, in order to ease comparisons with the actual scheme of revenue sharing, we also provide the percentage levels obtained by each team under both rules.

TEAMS	$S_i(N, A)$	%	$CD_i(N, A)$	%
RM	$78656,39$	$25,52$	$148926,845$	$48,31$
BCN	$52059,43$	$16,89$	$92777,707$	$30,10$
ATM	$26361,17$	$8,55$	$38525,825$	$12,50$
SVQ	$12385,91$	$4,02$	$9022,498$	$2,93$
BET	$11768,03$	$3,82$	$7718,085$	$2,50$
VAL	$11076,05$	$3,59$	$6257,238$	$2,03$
CEL	$11037,48$	$3,58$	$6175,813$	$2,00$
DPV	$9967,97$	$3,23$	$3917,958$	$1,27$
ATH	$9239,32$	$3,00$	$2379,697$	$0,77$
RVL	$9005,43$	$2,92$	$1885,929$	$0,61$
RSC	$8639,68$	$2,80$	$1113,791$	$0,36$
VIL	$8570,33$	$2,78$	$967,385$	$0,31$
LPA	$8341,57$	$2,71$	$484,447$	$0,16$
SPO	$7936,87$	$2,57$	$-369,919$	$-0,12$
MLG	$7872,08$	$2,55$	$-506,698$	$-0,16$
GRA	$7785,63$	$2,53$	$-689,204$	$-0,22$
EIB	$7491,51$	$2,43$	$-1310,124$	$-0,42$
ESP	$7304,55$	$2,37$	$-1704,817$	$-0,55$
LEV	$7297,33$	$2,37$	$-1720,059$	$-0,56$
GET	$5462,88$	$1,77$	$-5592,787$	$-1,81$

Table 3. The Shapley and concede-and-divide outcomes for the Spanish Football League in 2015/2016.

Several conclusions can be derived from Table 3. Maybe the most obvious one is that seven teams would be awarded negative values under the concede-and-divide rule. That is, they should be compensating the other teams (for an overall amount of almost 4% of the pie) because they are not bringing enough audiences on their own to the tournament, and they are somewhat benefitting from competing in this tournament.

As we can also see, and contrary to what some might argue, the actual revenue sharing seems to be biased against the two powerhouses. In particular, Real Madrid should be obtaining a quarter of the pie with the Shapley rule and almost half of it with the concede-and-divide rule. Barcelona would also go up (from 14.5% to almost 17% and 30%, respectively). Atlético de Madrid would increase considerably too. All the other teams would decrease, with the exception of Celta de Vigo and Real Betis Balompié (the greatest team on earth, at least according to one of the co-authors of this paper) who would increase if the Shapley rule would be implemented (but not if the concede-and-divide rule would be implemented).

Finally, under the Shapley rule, the two powerhouses would be obtaining (combined) slightly above 40% of the pie. Under the concede-and-divide rule, they would be obtaining a staggering 78.4%. The latter distribution, which also exhibits the feature of making seven teams pay (rather than receive), thus seems difficult to be accepted in this case. Nevertheless, if the real outcome is the result of a bargaining process among the participating teams, we cannot deny the fact that the two powerhouses have a very strong bargaining power, which might largely influence the final outcome.

Now, it has been argued that an extremely unequal sharing of the broadcasting revenues would be detrimental for the overall quality of the tournament. Some even go further claiming that a system with unequal shares of revenue, widening the gap between clubs, might violate EU competition law.[25] Clubs with higher earnings will be able to attract more playing talent. Eventually, this will make them prevail (overwhelmingly) in their national tournaments winning easily most of the games. Likewise, teams with lower earnings will become less competitive, eventually giving up while playing against the richest teams (preserving their key players for the ensuing more balanced games against other *peer* teams). This will render most of the games

[25] In late 2014, the so-called FASFE (an organization consisting of groups of fans, club members, and minority shareholders of several Spanish professional football clubs) and the International Soccer Centre (a movement that aims to obtain more balanced and transparent football and basketball competitions in Spain) filed an antitrust complaint with the European Commission against the Spanish National Professional Football League.

in the tournament uninteresting (even for the fan base of the rich teams). Because of this, one might argue that a sharing process based on performance might not be that different from a sharing process based on audiences. In the case of the data presented above, there is indeed a positive correlation between the ranking of teams according to performance (league scores) and TV audiences. Nevertheless, the cardinal aspect of both rankings differs substantially. More precisely, in terms of scoring, there is less than a 3 to 1 ratio between the first and the last team, whereas in terms of TV audiences, there is more than a 14 to 1 ratio between the first and the last team.

To account for the above (at least, partially) we consider alternative schemes with our database described above. More precisely, we present in what follows different mixed schemes in which a portion of the overall revenue is equally divided, another is proportionally divided according to scoring performance, and the residual is divided according to one of our two rules (thus, only taking into account the audiences). Note that this is indeed what happens in the most important European football leagues, as described in Table 4. La Liga itself has decided to implement a new scheme for the current season (2016-2017) in which, as shown by the table, half of the overall revenue will be shared equally, whereas one quarter will be shared according to performance and the remaining quarter according to audiences.

Criteria → Country	Egalitarian	League performance: Scoring, ...	Social performance: TV audience, ...
England	50%	25%	25%
Germany		100%	
Italy	40%	30%	30%
Spain (new)	50%	25%	25%

Table 4. Hybrid revenue sharing in the most important European football leagues.

Table 5 yields the allocations proposed by our two rules (Shapley and concede-and-divide) for the new budget shared among participating teams in La Liga in 2016/2017. It also shows the actual scheme that was approved by La Liga in the first column. Note that the overall revenue to be shared this season went up more than 30% from the previous season. Unfortunately, we do not have data on audiences for the 2016/2017 season yet. Thus, we simply replicate those from the 2015/2016 season.[26]

TEAMS	Revenues	%	$S_i(N, A)$	%	$CD_i(N, A)$	%
RM	145,0	$11,2$	$331,7$	$25,52$	$628,1$	$48,31$
BCN	152,5	$11,7$	$219,5$	$16,89$	$391,3$	$30,10$
ATM	102,0	$7,8$	$111,2$	$8,55$	$162,5$	$12,50$
SVQ	77,0	$5,9$	$52,2$	$4,02$	$38,0$	$2,93$
BET	47,0	$3,6$	$49,6$	$3,82$	$32,5$	$2,50$
VAL	90,5	$7,0$	$46,7$	$3,59$	$26,4$	$2,03$
CEL	90,5	$7,0$	$46,5$	$3,58$	$26,0$	$2,00$
DPV	44,0	$3,4$	$42,0$	$3,23$	$16,5$	$1,27$
ATH	73,5	$5,7$	$39,0$	$3,00$	$10,0$	$0,77$
RVL	45,5	$3,5$	$38,0$	$2,92$	$8,0$	$0,61$
RSC	59,0	$4,5$	$36,4$	$2,80$	$4,7$	$0,36$
VIL	53,5	$4,1$	$36,1$	$2,78$	$4,1$	$0,31$
LPA	40,0	$3,1$	$33,5$	$2,71$	$2,0$	$0,16$
SPO	42,5	$3,3$	$35,2$	$2,57$	$-1,6$	$-0,12$
MLG	52,0	$4,0$	$33,2$	$2,55$	$-2,1$	$-0,16$
GRA	44,5	$3,4$	$32,8$	$2,53$	$-2,9$	$-0,22$
EIB	40,5	$3,1$	$31,6$	$2,43$	$-5,5$	$-0,42$
ESP	50,0	$3,8$	$30,8$	$2,37$	$-7,2$	$-0,55$
LEV	49,5	$3,8$	$30,8$	$2,37$	$-7,3$	$-0,56$
GET	46,5	$3,6$	$23,0$	$1,77$	$-23,6$	$-1,81$

Table 5. The revenues, as well as the Shapley and concede-and-divide outcomes for the Spanish Football League in 2016/2017.

At the risk of stressing the obvious, Table 5 does not yield qualitative differences with respect to Table 3. Numbers are essentially rescaled up by the increase in the overall budget and, therefore, percentages (both for the Shapley and concede-and-divide rules) remain the

[26]This implies, in particular, that we consider the same teams, albeit three (Getafe, Rayo Vallecano and Levante) were relegated at the end of the 2015/2016 season and replaced by three other teams from the second division for the 2016/2017 season (Alavés, Leganés and Osasuna).

same.

We conclude by considering the hybrid scheme mentioned in Table 4 for La Liga in 2016/2017. More precisely, we assume that half of the overall revenue will be shared equally (that would represent 32,5 million euros for each team), whereas one quarter will be shared according to performance and the remaining quarter according to our two rules. By performance, La Liga refers to the aggregate scores in the previous five seasons (where a zero score is given to those teams that played in the second division, or below, in one of those years). One quarter of the budget is then allocated proportionally to those 5-year scores. The Shapley and concede-and-divide columns of Table 6 are the result of aggregating (for each team) the fixed amount (32,5 million), the amount proportional to performance, and the amount suggested by the corresponding rule for the division of the remaining quarter of the budget.

TEAMS	Revenues	%	$S_i(N, A)$	%	$CD_i(N, A)$	%
RM	145,0	$11,2$	$162,9$	$12,5$	$237,0$	$18,2$
BCN	152,5	$11,7$	$142,4$	$11,0$	$185,3$	$14,3$
ATM	102,0	$7,8$	$102,8$	$7,9$	$115,6$	$8,9$
SVQ	77,0	$5,9$	$75,6$	$5,8$	$72,0$	$5,5$
BET	47,0	$3,6$	$48,9$	$3,8$	$44,6$	$3,4$
VAL	90,5	$7,0$	$79,2$	$6,1$	$74,1$	$5,7$
CEL	90,5	$7,0$	$49,1$	$3,8$	$44,0$	$3,4$
DPV	44,0	$3,4$	$46,5$	$3,6$	$40,1$	$3,1$
ATH	73,5	$5,7$	$65,2$	$5,0$	$58,0$	$4,5$
RVL	45,5	$3,5$	$48,5$	$3,7$	$41,0$	$3,2$
RSC	59,0	$4,5$	$58,1$	$4,5$	$50,2$	$3,9$
VIL	53,5	$4,1$	$49,5$	$3,8$	$41,5$	$3,2$
LPA	40,0	$3,1$	$42,3$	$3,3$	$34,0$	$2,6$
SPO	42,5	$3,3$	$43,4$	$3,3$	$34,6$	$2,7$
MLG	52,0	$4,0$	$52,3$	$4,0$	$43,5$	$3,3$
GRA	44,5	$3,4$	$46,2$	$3,6$	$37,3$	$2,9$
EIB	40,5	$3,1$	$41,9$	$3,2$	$32,6$	$2,5$
ESP	50,0	$3,8$	$49,2$	$3,8$	$39,7$	$3,1$
LEV	49,5	$3,8$	$50,2$	$3,9$	$40,7$	$3,1$
GET	46,5	$3,6$	$45,8$	$3,5$	$34,1$	$2,6$

Table 6. The revenues, as well as the hybrid Shapley and concede-and-divide outcomes for the Spanish Football League in 2016/2017.

The first novelty we observe from the data at Table 6 is that no team is awarded a negative amount under the (hybrid) concede-and-divide scheme. Although the concede-and-divide rule indeed suggests negative values for seven teams (as mentioned above for the whole allocation of the budget, which also remains true for only one quarter of it), this is compensated by the fixed (equal) amount each team obtains in this hybrid scheme.

Another obvious aspect is that the hybrid schemes become more egalitarian. Under the full-fledged Shapley rule, the two powerhouses were obtaining (combined) slightly above 40%

of the pie. The hybrid scheme lowers this to 23.5%. Under the full-fledged concede-and-divide rule, they were obtaining a staggering 78.4%, which now moves down (under the hybrid scheme) to 32.5% (less than half).

A final aspect is that the two hybrid schemes produce much more similar allocations to the existing one. Especially, in the case of the Shapley (hybrid) scheme, for which only two teams vary more than 1% with respect to the existing scheme (Real Madrid from $11,2\%$ to 12.5% and Celta de Vigo from 7% to 3.8%). One might then argue, with the caveat on audience numbers mentioned above, that the *hybrid Shapley* rule provides rationale for the current scheme being implemented in La Liga

7 Discussion

We have presented a stylized model to deal with the problem of sharing the revenues from broadcasting sports events. We have provided normative, empirical and strategic foundations for rules sharing each game's revenues equally or proportionally among the participating teams. Both rules have distinguishing merits. One (the concede-and-divide rule) is supported by an intuitive procedure aiming to reflect the (potentially different) fan base of each team. Another (the Shapley rule) is supported by a powerful (and normatively appealing) stability property preventing secessions from participating players.

We have also provided as a case study an empirical application deriving what both rules would suggest for the Spanish Football League (La Liga). Our results largely differ from the schemes that were traditionally used, which we find (somewhat surprisingly) biased against the three teams driving the largest audiences. Hybrid schemes in which our rules are only used to share one fourth of the budget, whereas another fourth is allocated according to performance, and the rest half is equally split seem to be closer to the current scheme being implemented by the Spanish National Professional Football League.

It is left for further research to enrich the model in plausible ways. For instance, some games are offered for free (in non-subscription channels), instead of pay per view. That might influence the audience numbers. In our case study (La Liga), not all teams are broadcasted under that option. And its broadcasting rights are negotiated independently. Thus, it might well make sense to talk about two different budgets: one coming from subscription channels (to which all team have access) and another coming from non-subscription channels (to which not all team have access, and which might be associated to different audience figures).

Similarly, several games might be broadcasted simultaneously, which might reduce the number of viewers for some games. And if all games are broadcasted in exclusive time windows (as it happens, for instance, in our case study), prime time is only awarded to some games. All these aspects might have an important impact on audience figures, which has been ignored in our model.

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