A Shapley Value for Cooperative Games with Hierarchies and Communication Restrictions

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Introduction I

Goal:
Understanding organizations with hierarchical as well as communication structures.

Restricted communication: Myerson (1977).
Only connected coalitions in a (communication) graph are feasible.

Hierarchy: Various models such as permission structures (using digraphs) and antimatroids.

Introduction II

Outline:

1. Introduction
2. Preliminaries:
   Restricted communication and hierarchies
3. Accessible union stable networks
4. Cooperative games on accessible union stable networks and axiomatizations of the Shapley value
5. Concluding remarks

Preliminaries: TU-games I

1. Cooperative Games with Transferable Utility

A **cooperative TU-game** is a pair (N, v) with

- $N = \{1, \ldots, n\}$ is a set of **players** (finite)
- $v \colon 2^N \to \mathbb{R}$ is a **characteristic function** satisfying $v(\emptyset) = 0$

The *worth* $v(S) \in \mathbb{R}$ is what the players in **coalition** $S \subseteq N$ can earn by cooperation.

In a TU-game it is assumed that all coalitions $S \subseteq N$ are feasible.

Usually we encounter restrictions in coalition formation, for example communication or hierarchical restrictions.

Restricted cooperation: $\mathcal{F} \subseteq 2^N$ is the set of feasible coalitions

Restricted communication I

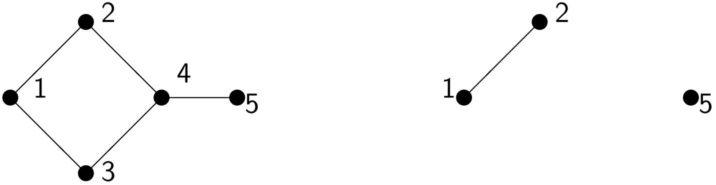
<u>2A. Restricted communication</u> (Myerson, 1977)
A **communication graph** is a pair (N, L) with

- $N = \{1, ..., n\}$ a finite set of **nodes** (players)
- $L \subseteq \{\{i, j\} \mid i, j \in N,\ i \neq j\}$ a set of **links**

A tuple (v, L) is a **(communication) graph game**

Feasible coalitions in L
$\mathcal{F}$: set of connected coalitions in (N, L).

Restricted communication II



The components of $\{1,2,5\}$ are $\{1,2\}$and $\{5\}$.

$v^L(\{1,2,5\}) = v(\{1,2\}) + v(\{5\})$.

Restricted communication III

Every coalition S can be partitioned into maximally connected coalitions (**components**) in $(S, L(S))$.

Restricted games

The **Myerson restricted game** v^L assigns to every coalition the sum of the worths of its components.

Solutions

We can apply any TU-game solution to the restricted game v^L, for example
the **Myerson value** assigns to every graph game the Shapley value of the restricted game: $\mu(v, L) = Sh(v^L)$.

Restricted communication IV

The **Shapley value** (Shapley, 1953):

$$Sh_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i^{\pi}(v) \text{ for all } i \in N,$$

where $\Pi(N)$ is the collection of all **permutations** of N and for $\pi \in \Pi(N)$

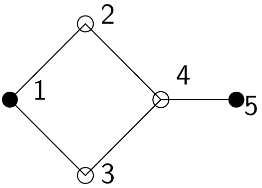
$$\begin{aligned} m_j^{\pi}(v) &= v(\{g \in N \mid \pi(g) \leq \pi(j)\}) \\ &\quad - v(\{g \in N \mid \pi(g) < \pi(j)\}) \end{aligned}$$

is the **marginal contribution** of player i to the coalition of all its

predecessors in permutation π.

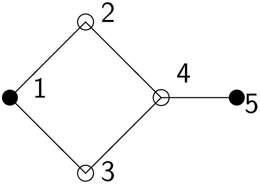
Restricted communication V

Example



$$v(S) = \begin{cases} 1 & \text{if } \{1,5\} \subseteq S \\ 0 & \text{else,} \end{cases}$$

Restricted communication VI



$$v^L(S) = \begin{cases} 1 & \text{if } S \in \{\{1,2,4,5\}, \{1,3,4,5\}, \{1,2,3,4,5\}\} \\ 0 & \text{else,} \end{cases}$$

$$\mu(v, L) = (\frac{3}{10}, \frac{1}{20}, \frac{1}{20}, \frac{3}{10}, \frac{3}{10}).$$

Restricted communication VII

Myerson (1977) axiomatized his value by *component efficiency* and *fairness*.

Component efficiency: every maximally connected set of players (component) earns exactly its worth.

Fairness: deleting (or adding) a link has the same effect on the payoffs of the two players on that link.

Games with a permission structure I

2B1. Games with a permission structure

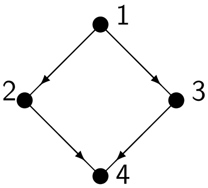
Gilles, Owen and van den Brink (1992)
van den Brink and Gilles (1996)
Gilles and Owen (1994)
van den Brink (1997, 1999, 2010)

A **permission structure** is a **digraph** (N, D) with

- $N = \{1, ..., n\}$ a finite set of **nodes** (players)
- $D \subseteq N \times N$ a **binary relation** on N

A pair (v, D) is a **game with a permission structure**

Games with a permission structure II



The conjunctive feasible coalitions are:
$\{1\}$, $\{1,2\}$, $\{1,3\}$, $\{1,2,3\}$ and $\{1,2,3,4\}$.

Additional disjunctive feasible coalitions:
$\{1,2,4\}$ and $\{1,3,4\}$.

Games with a permission structure III

Conjunctive feasible coalitions in D

Those coalitions such that for every player in the coalition *all* its (direct) predecessors belong to the coalition.

Disjunctive feasible coalitions in D

.... *at least one* .... (except the top players)

Remark: In both sets of feasible coalitions, every coalition has a unique largest feasible subset.
(This follows from union closedness.)

Games with a permission structure IV

Two restricted games

The **conjunctive restriction** $r^c_{v,D}$ of v on D assigns to every coalition the worth of its largest conjunctive feasible subset.

Similar, we define the **disjunctive restriction** $r^d_{v,D}$ which assigns to every coalition the worth of its largest disjunctive feasible subset.

Solutions

We can apply any TU-game solution to the restricted games, for example the Shapley value:

$\varphi^c(v, D) = Sh(r^c_{v,D})$ and $\varphi^d(v, D) = Sh(r^d_{v,D})$

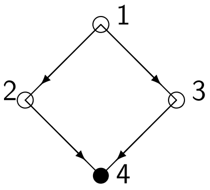
Games with a permission structure V

Example
$N = \{1, 2, 3, 4\}$

$$v(S) = \begin{cases} 1 & \text{if } S \ni 4 \\ 0 & \text{else,} \end{cases}$$

$$D = \{(1,2), (1,3), (2,4), (3,4)\}$$

Games with a permission structure VI

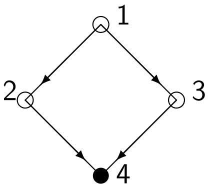


Then

$$r^c_{v,D}(S) = \begin{cases} 1 & \text{if } S = \{1,2,3,4\} \\ 0 & \text{else} \end{cases}$$

$$\varphi^c(v,D) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$$

Games with a permission structure VII



$$r^d_{v,D}(S) = \begin{cases} 1 & \text{if } S \in \{\{1,2,4\},\ \{1,3,4\},\ \{1,2,3,4\}\} \\ 0 & \text{else} \end{cases}$$

$$\varphi^d(v,D) = (\frac{5}{12}, \frac{1}{12}, \frac{1}{12}, \frac{5}{12})$$

Games with a permission structure VIII

The conjunctive and disjunctive approach coincide if the permission structure is a rooted tree.

Special cases: auction games, dual airport games, polluted river games, hierarchically structured firms, joint liability games ...

Results on:
Game properties
Harsanyi dividends
Axiomatizations of solutions (a.o. with conjunctive, respectively disjunctive, fairness)
Computation of solutions

Games on antimatroids I

2B2. Cooperative games on antimatroids

Dillworth (1940), Algaba, Bilbao, van den Brink and Jiménez-Losada (2003, 2004a, 2004b)

Definition A set of feasible coalitions $\mathcal{A} \subseteq 2^N$ is an **antimatroid** on N if it satisfies

1. $\emptyset \in \mathcal{A}$
2. (Closed under union) If $S, T \in \mathcal{A}$ then $S \cup T \in \mathcal{A}$
3. (Accessibility) If $S \in \mathcal{A}$, $S \neq \emptyset$, then there exists an $i \in S$ such that $S \setminus \{i\} \in \mathcal{A}$.

An antimatroid $\mathcal{A} \subseteq 2^N$ is **normal** if for every $i \in N$ there exists an $S \in \mathcal{A}$ such that $i \in S$.

Games on antimatroids II

Theorem (Algaba et al., 2004a)
If D is an acyclic permission structure on N then the conjunctive and disjunctive feasible sets are normal antimatroids on N.

Remark: The set of feasible coalitions in *ordered partition voting* cannot be obtained from (conjunctive or disjunctive) permission structures.

Games on antimatroids III

Example Suppose the player set $N = \{1, 2, 3, 4, 5\}$ is partitioned in two levels:

Level 1: players 1, 2 and 3

Level 2: players 4 and 5

Feasible coalitions: every subset of level 1, and every subset of level 2 with a majority of level 1.

The set of feasible coalitions is an antimatroid

but it is not a poset antimatroid

and it does not satisfy the path property

Games on antimatroids IV

Example

$$\mathcal{A} = \left\{ \begin{array}{l} \emptyset, \{1\}, \{2\}, \{3\}, \\ \\ \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \\ \\ \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \\ \\ \{2,3,4\}, \{2,3,5\}, \\ \\ \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,3,4,5\} \end{array} \right\}$$

Games on antimatroids V

Theorem (Algaba et al., 2004a)
Let $\mathcal{A}$ be an antimatroid. There is an acyclic permission structure D such that $\mathcal{A}$ is its conjunctive feasible set if and only if $\mathcal{A}$ is *closed under intersection* (i.e. $S \cap T \in \mathcal{A}$ for every $S, T \in \mathcal{A}$).

Remark: Poset antimatroids

Remark: Disjunctive feasible sets and the path property

Games on antimatroids VI

Definitions

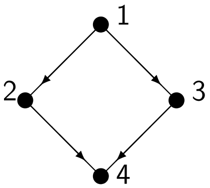
Player $i \in S \in \mathcal{A}$ is an **extreme player** of S if $S \setminus \{i\} \in \mathcal{A}$.

$S \in \mathcal{A}$ is a **path** in $\mathcal{A}$ if it has a unique extreme player.

The path $S \in \mathcal{A}$ is a ***i*-path** if it has $i \in S$ as unique extreme player.

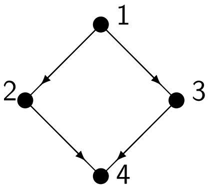
Remark: The paths form the 'basis' of the antimatroid, i.e. every feasible coalition $S \in \mathcal{A}$ is either a path or a union of paths.

Games on antimatroids VII



The conjunctive feasible coalitions are:
$\underline{\{1\}}$, $\underline{\{1,2\}}$, $\underline{\{1,3\}}$, $\{1,2,3\}$ and $\underline{\{1,2,3,4\}}$.

Games on antimatroids VIII



The conjunctive feasible coalitions are:
$\underline{\{1\}}$, $\underline{\{1,2\}}$, $\underline{\{1,3\}}$, $\{1,2,3\}$ and $\{1,2,3,4\}$.

Additional disjunctive feasible coalitions:
$\underline{\{1,2,4\}}$ and $\underline{\{1,3,4\}}$.

Games on antimatroids IX

Theorem (Algaba et al., 2003)
Let $\mathcal{A}$ be an antimatroid. There is an acyclic permission structure D such that $\mathcal{A}$ is its conjunctive feasible set if and only if every player $i \in N$ has a unique i-path in $\mathcal{A}$.

Theorem (Algaba et al., 2003)
Let $\mathcal{A}$ be an antimatroid. There is an acyclic permission structure D such that $\mathcal{A}$ is its disjunctive feasible set if and only if $\mathcal{A}$ is an antimatroid satisfying the path property.

Games on antimatroids X

Definition

An antimatroid $\mathcal{A}$ satisfies the **path property** if it satisfies the following conditions:

1. Every path S has a unique feasible ordering, i.e. $S := (i_1 > \cdots > i_t)$ such that $\{i_1, \ldots, i_k\} \in \mathcal{A}$ for all $1 \leq k \leq t$. Furthermore, the union of these orderings for all paths is a partial ordering of N.
2. If S, T and $S \setminus \{i\}$ are paths such that the extreme players of T equals the extreme player of $S \setminus \{i\}$, then $T \cup \{i\} \in \mathcal{A}$.

Games on antimatroids XI

(Korte, Lóvasz and Schrader, 1991)

The **interior operator** $int_{\mathcal{A}} \colon 2^N \to \mathcal{A}$ is given by

$$int_{\mathcal{A}}(S) = \bigcup \{T \in \mathcal{A} \mid T \subseteq S\} \text{ for all } S \subseteq N$$

Definition

The **restriction** of game v on antimatroid $\mathcal{A}$ is the game $v_{\mathcal{A}}$ given by

$$v_{\mathcal{A}}(S) = v(int_{\mathcal{A}}(S)) \text{ for all } S \subseteq N$$

Restricted Shapley value:

$\varphi^{Sh}(v, \mathcal{A}) = Sh(v_{\mathcal{A}})$

Games on antimatroids XII

Remarks:

1. Axiomatic characterizations of the restricted Shapley value on the class of games on antimatroids.

2. Characterizing subclasses of antimatroids by characterizing properties of the restricted Shapley value.

see Algaba, Bilbao, van den Brink and Jiménez-Losada (2003)

2B3. Communication versus hierarchies

Let $\mathcal{F} \subseteq 2^N$ be the set of connected coalitions in some (undirected) communication graph.

Then $\mathcal{F}$ is not closed under union, but satisfies the weaker **union stability**, see Algaba, Bilbao, Borm and López (2000, 2001).

$\mathcal{F}$ does satisfy accessibility. It even satisfies the stronger **2-accessibility**.

Communication versus hierarchies II

Theorem (van den Brink, 2012)
Let $\mathcal{F} \subseteq 2^N$. Then $\mathcal{F}$ is the set of connected coalitions in some (undirected) communication graph if and only if it satisfies

1. $\emptyset \in \mathcal{F}$
2. (Union stability) If $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$ then $S \cup T \in \mathcal{F}$
3. (2-Accessibility) If $S \in \mathcal{F}$, $|S| \geq 2$, then there exist $i, j \in S$, $i \neq j$, such that $S \setminus \{i\}, S \setminus \{j\} \in \mathcal{F}$
4. (Normality) For every $i \in N$ there is an $S \in \mathcal{F}$ such that $i \in S$.

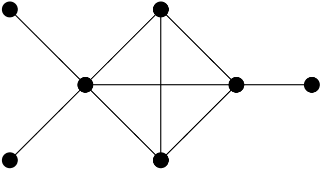
Communication versus hierarchies III

Adding other properties characterizes the sets of connected coalitions in special graphs, for example adding *closedness under intersection* yields those arising from *cycle-complete communication graphs*.

Other special cases: line-graphs, cycle-free graphs

A graph is cycle-complete if, whenever there is a cycle, the subgrah on that cyce is complete.

Communication versus hierarchies IV



Communication versus hierarchies V

<u>Antimatroids:</u>		<u>Communication:</u>
$\emptyset$ is feasible		$\emptyset$ is feasible
union closed	$\Rightarrow$	union stable
accessible	$\Leftarrow$	2-accessible

Accessible union stable networks I

3A. Next, we take the weaker of the two pairs (union and accessibility) properties.

Definition A normal set of feasible coalitions $\mathcal{F} \subseteq 2^N$ is an **accessible union stable network system** on N if it satisfies

1. $\emptyset \in \mathcal{F}$
2. (Union stability) If $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$ then $S \cup T \in \mathcal{F}$
3. (Accessibility) If $S \in \mathcal{F}$, $S \neq \emptyset$, then there exists an $i \in S$ such that $S \setminus \{i\} \in \mathcal{F}$.

Remark: These set network systems contain all communication feasible sets and antimatroids.

Accessible union stable networks II

Proposition
If $\mathcal{F}$ is an augmenting system then $\mathcal{F}$ is an accessible union stable system.

Augmenting systems are introduced by Bilbao (2003).

Definition A set of feasible coalitions $\mathcal{F} \subseteq 2^N$ is an **augmenting system** on N if it satisfies

1. $\emptyset \in \mathcal{F}$
2. (Union stability) If $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$ then $S \cup T \in \mathcal{F}$
3. (Augmentation 1) If $S, T \in \mathcal{F}$ with $S \subset T$, then there exists an $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.

Accessible union stable networks III

Remark: The reverse is not true.

So, under union stability

Augmentation $\Rightarrow$ Accessibility

but not the other way around.

Accessible union stable networks IV

Example

Consider two societies, say N and M. Every subset of each society can form a feasible coalition.

Further, every subset of society N can form a coalition 'outside' N, but the players in M can only form a coalition with 'outside' players when they join all together.

So, we can consider society N as 'explorers' and society M as a 'careful' society.

The corresponding set of feasible coalitions is an accessible union stable network.

Accessible union stable networks V

It cannot be the set of connected coalitions of a communication graph (since it does not satisfy 2-accessibility).

It is not an antimatroid (since it is not closed under union).

It is not an augmenting system (since it does not satisfy Augmentation 1).

Accessible union stable networks VI

Example: Two societies

$N = \{1, 2\}$ (Explorers)

$M = \{3, 4, 5\}$ (Careful players)

$$\mathcal{F} = \left\{ \begin{array}{l} \emptyset, \\ \{1\}, \{2\}, \{1,2\}, \\ \{3\}, \{4\}, \{5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{3,4,5\}, \\ \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\} \end{array} \right\}$$

Accessible union stable networks VII

Proposition

Let $\mathcal{F} \subseteq 2^N$ be a set system. If $\mathcal{F}_i = \{T \in \mathcal{F} \mid i \in T\}$ is an antimatroid for all $i \in N$, then $\mathcal{F}$ is an accessible union stable network.

Remark: The reverse is not true.

Example $N = \{1, 2, 3\}$ and

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{2, 3\}, N\}.$$

This is an accessible union stable network but

$$\mathcal{F}_1 = \{\{1\}, N\}.$$

Dual structures I

3B. Dual structures

The dual structure of $\mathcal{F} \subseteq 2^N$ is

$$\mathcal{F}^d = \{S \subseteq N \mid N \setminus S \in \mathcal{F}\}$$

It is known that when $\mathcal{A}$ is an antimatroid, then $\mathcal{A}^d$ is a convex geometry.

Dual structures II

Convex geometries are introduced by Edelman and Jamison (1985).

Definition A set of feasible coalitions $\mathcal{F} \subseteq 2^N$ is a *convex geometry* on N if it satisfies

1. $\emptyset \in \mathcal{F}$
2. (Intersection closed) If $S, T \in \mathcal{F}$ then $S \cap T \in \mathcal{F}$
3. (Augmentation 2) If $S \in \mathcal{F}$ with $S \neq N$, then there exists an $i \in N \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.

Remark: Every convex geometry is normal.

Dual structures III

Proposition

Let $\mathcal{F}$ be a set system with $N \in \mathcal{F}$. Then $\mathcal{F}$ is an accessible union stable network if and only if $\mathcal{F}^d$ satisfies

1. $\emptyset \in \mathcal{F}$
2. (Weak intersection closed) If $S, T \in \mathcal{F}$ with $S \cup T \neq N$, then $S \cap T \in \mathcal{F}$
3. (Augmentation 2) If $S \in \mathcal{F}$ with $S \neq N$, then there exists an $i \in N \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.

Dual structures IV

Remark: Every convex geometry satisfies these properties.

The reverse is not true.

Example $N = \{1, 2, 3\}$ and

$$\mathcal{F}^d = \{\emptyset, \{1\}, \{1, 3\}, \{2, 3\}, N\}$$

Supports I

3C. Supports

Let $\mathcal{F}$ be a union stable system. Then $S \in \mathcal{F}$ is a **support** of $\mathcal{F}$ if and only if S is not the union of two non-disjoint feasible coalitions.

So, in a union stable system a feasible coalition is either a support or a union of at least two feasible coalitions.

Remark: The supports of a communication feasible set are the singletons and the edges.

Supports II

Proposition
Let $\mathcal{F}$ be an accessible union stable network.
If $S \in \mathcal{F}$ is a support of $\mathcal{F}$ with $|S| \geq 2$ then S is a path of $\mathcal{F}$.

However, not every path with at least two players is a support of $\mathcal{F}$.

Example $N = \{1, 2, 3, 4\}$ and

$$\mathcal{F} = \{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{3, 4\}, \{2, 3, 4\}, \underline{\{1, 2, 3, 4\}}\}.$$

Cooperative games on accessible union stable networks I

4A. Cooperative games on accessible union stable networks

We consider the restricted game on union stable systems of Algaba, Bilbao, Borm and López (2000, 2001).

Consider the set of supports of a union stable system.

By repeatedly taking the 'closure under union stability' from step k (k finite) the set does not change anymore, and is a partition of the player set N.

The elements of this partition are called the **components** in the union stable system.

The components of coalition S in a normal union stable system form a partition of S.

$C_{\mathcal{F}}(S)$: set of components of $S \subseteq N$.

Cooperative games on accessible union stable networks II

$(v, \mathcal{F})$: game on an accessible union stable network.

$GAUS^N$: collection of all games on an accessible union stable network.

The restricted game

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T) \text{ for all } S \subseteq N.$$

The restricted Shapley value:

$$\varphi(v, \mathcal{F}) = Sh\left(v^{\mathcal{F}}\right).$$

Axiomatization I

4B. Axiomatization

Component efficiency: If M is a component of $\mathcal{F}$ then $\sum_{i \in M} f_i(v, \mathcal{F}) = v(M)$.

Player $i \in N$ is a **component dummy** in $\mathcal{F}$ if $M \in \mathcal{F} \Rightarrow i \notin M$.

Component dummy: If i is a component dummy in $\mathcal{F}$ then $f_i\left(v, \mathcal{F}\right) = 0$.

Axiomatization II

For $\mathcal{F} \subseteq 2^N$ and $i \in N$, define

$$\mathcal{F}_{-i} = \{S \in \mathcal{F} \mid i \notin S\}.$$

Proposition

If $\mathcal{F} \subseteq 2^N$ is an accessible union stable network and $i \in N$, then $\mathcal{F}_{-i}$ is an accessible union stable network.

Balanced contributions

For every $(v, \mathcal{F}) \in GAUS^N$ and any two players $i, j \in N$ with $i \neq j$, it holds that

$$f_i(v, \mathcal{F}) - f_i(v, \mathcal{F}_{-j}) = f_j(v, \mathcal{F}) - f_j(v, \mathcal{F}_{-i}).$$

Axiomatization III

Theorem

The Shapley value is the unique value on the class $GAUS^N$ that satisfies component efficiency, component dummy, and has balanced contributions.

Axiomatization IV

Instead of considering the effects of deleting all coalitions containing a particular player on the payoff of another player, we can also consider the effect when deleting all coalitions containing both players.

For an accessible union stable network $\mathcal{F}$ and two players $i, j \in N$, $i \neq j$, define

$$\mathcal{F}_{-ij} = \{S \in \mathcal{F} \mid \{i, j\} \not\subseteq S\}.$$

Fairness

For every $(v, \mathcal{F}) \in GAUS^N$ and $i, j \in N$ such that $\mathcal{F}_{-ij}$ is an accessible union stable network, it holds that

$$f_i(v, \mathcal{F}) - f_i(v, \mathcal{F}_{-ij}) = f_j(v, \mathcal{F}) - f_j(v, \mathcal{F}_{-ij})$$

Axiomatization V

Remark: The restriction that $\mathcal{F}_{-ij}$ is an accessible union stable network implies that not all feasible coalitions can be deleted.

Proposition

If $\mathcal{F} \subseteq 2^N$ is an accessible set system then $\mathcal{F}_{-ij}$ is accessible.

However, for an arbitrary accessible union stable network $\mathcal{F}$ the set system $\mathcal{F}_{-ij}$ need not be union stable.

Axiomatization VI

Example Consider

$$\mathcal{F} = \left\{ \begin{array}{l} \emptyset, \\ \{1\}, \{2\}, \{1,2\}, \\ \{3\}, \{4\}, \{5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{3,4,5\}, \\ \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\} \end{array} \right\}$$

Take a player from N and one from M, for example players 2 and 4. Then $\mathcal{F}_{-24} = \mathcal{F} \setminus \{\{2,3,4,5\}, \{1,2,3,4,5\}\}$, which is not union stable since $\{1,2\}$ and $\{1,3,4,5\}$ both belong to $\mathcal{F}_{-24}$ but their union does not.

Axiomatization VII

Union stability is kept in the following subclass introduced in Algaba, Bilbao, Borm and López (2001).

Definition An accessible union stable network system $\mathcal{F} \subseteq 2^N$ belongs to the subclass $AUSI^N$ if

(1) it is *2-intersection closed*, i.e. for all $S, T \in \mathcal{F}$ with $|S \cap T| \geq 2$, we have $S \cap T \in \mathcal{F}$,

(2) it is *cycle-free*, i.e. every non-unitary feasible coalition can be written in a unique way as a union of non-unitary supports.

Axiomatization VIII

Remark: This subclass contains the class of connected coalitions in a cycle-free communication graph and poset antimatroids.

Applications in, e.g. auction situations (see Graham, Marshall and Richard (1990)), airport games (see Littlechild and Owen (1973)), sequencing games (see Curiel, Potters, Rajendra Prasad, Tijs and Veltman (1993, 1994)), water distribution problems (see Ambec and Sprumont (2002)) or polluted river problems (see Ni and Wang (2007)).

Axiomatization IX

Proposition

If

(i) $\mathcal{F} \in AUSI^N$, and

(ii) there is a support H such that $\{i, j\} \subseteq H$,

then $\mathcal{F}_{-ij}$ is union stable.

Proposition

If $\mathcal{F} \in AUSI^N$ and there is a support H containing players $i, j \in N$ then $\mathcal{F}_{-ij} \in AUSI^N$.

Theorem

The Shapley value is the unique value on $GAUSI^N$ that satisfies component efficiency, component dummy and fairness.

Cycle-free set systems I

Cycle-free set systems

We call set system $\mathcal{F} \subseteq 2^N$ **cycle-free** if all non-unitary feasible coalitions can be written in a unique way as a union of non-unitary supports.

Proposition

If there is an undirected graph L such that $\mathcal{F}$ is the set of connected coalitions in L, then
[$\mathcal{F}$ is cycle free if and only if L is cycle-free].

Cycle-free set systems II

Corollary

If $\mathcal{F}$ is a cycle-free communication feasible set then $\mathcal{F}$ is 2-intersection closed.

This does not hold the other way around, as illustrated by $\mathcal{F} = 2^N$ which is 2-intersection closed but not cycle-free.

Also, this proposition does not hold for arbitrary union stable systems, as illustrated by

$$\mathcal{F} = \{\emptyset, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}.$$

Concluding remarks I

5. Concluding remarks and some future research questions:

1. Relations among different models: What can they 'learn' from each other?

2. Characterizing set systems by axiomatizations of game solutions

3. Computation

4. Applications

Concluding remarks II

<u>5. Coalitional fairness:</u> If $S \in \mathcal{F}$ is such that $\mathcal{F} \setminus \{S\}$ is an accessible union stable network system then, for all $i, j \in S$, it holds that

$$f_i(N, v, \mathcal{F}) - f_i(N, v, \mathcal{F} \setminus \{S\}) = f_j(N, v, \mathcal{F}) - f_j(N, v, \mathcal{F} \setminus \{S\})$$

Algaba, Bilbao, Borm and López (2001): Shapley value for union stable systems (Myerson value) is characterized by component eff. and fairness.

van den Brink (1997): two similar axioms do not characterize the Shapley value for games with a permission structure.

Algaba, Bilbao, van den Brink and Jimenez-Lósada (2003): also such axioms do not characterize the Shapley value for games on normal antimatroids.

Concluding remarks III

How does it look for games on accessible union stable network systems?

We use the same component efficiency as above.

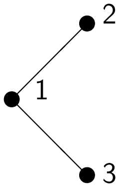
To apply coalitional fairness, we can only delete feasible coalitions that are supports that are not covered by a path.

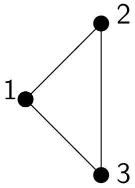
Definition Coalition $S \in \mathcal{F}$ is **covered by a path** if there is a path $T \in \mathcal{F}$ such that $S \subset T$ and $|T| = |S| + 1$.

Concluding remarks IV

6. Network formation

7. Value of a graph (Jackson and Wolinsky, 1996)





Concluding remarks V

8. A structure that does not give an accessible union stable structure.

Consider (N, D) with $D \subset N \times N$ a permission structure.

Set of feasible coalitions: those coalitions that are (weakly) connected and conjunctive feasible. These systems need not satisfy accessibility.

Example Consider $N = \{1, 2, 3, 4, 5\}$ and

$$D = \{(1, 2), (2, 3), (5, 4), (4, 3)\}.$$

Then N has no extreme players.

Concluding remarks VI